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HARMONIC ANALYSIS ON VECTOR BUNDLES OVER $\mathrm{Sp}(1,n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$

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ABSTRACT. Harmonic analysis on vector bundles over $\mathrm{Sp}(1,n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ associated with a finite dimensional representation τ of $\mathrm{Sp}(1)$ is developed using Godement's approach of trace spherical functions. The trace spherical trace functions are written in terms of Jacobi functions, and among them the positive definite ones are singled out. An inversion formula for the generalized Abel transform is given explicitly. The Paley-Wiener theorem, the inversion formula and the Plancherel theorem for the τ-spherical transform are determined.

INTRODUCTION

Harmonic analysis over Riemannian symmetric spaces of noncompact type is a fundamental and powerful area of mathematics that exhibits a beautiful interplay between the theory of special functions and the representation theory of semisimple Lie groups. Grown around the monumental work of Harish-Chandra, it has nowadays reached a nearly complete formulation, but, in its development, it has also laid the foundations of its natural extension: harmonic analysis on vector bundles over Riemannian symmetric spaces of noncompact type. Motivated also by many physical applications, this new subject is currently studied very intensively (cf. for instance [BR], [O], [Shi], [Cam], [P], [vdV], [M], [Dei], [BOS]), but a general theory has not yet been formulated.

In this paper we present a complete treatment of harmonic analysis for the spherical transform on a certain class of vector bundles over the hyperbolic space $\mathrm{Sp}(1,n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$. Set $G = \mathrm{Sp}(1,n)$ and $K = \mathrm{Sp}(1) \times \mathrm{Sp}(n)$. The class of vector bundles we consider are those associated with finite-dimensional irreducible representations τ of K which are trivial on $\mathrm{Sp}(n)$, so actually finite dimensional representations of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. This setting is sufficiently general to exhibit the new features of the theory, namely representations τ of arbitrary dimensions and the possible occurrence of the discrete series in the Plancherel formula. But, at the same time, it is also very concrete and therefore allows us to determine very explicit formulas, which make this paper also a ready-to-use reference for applications of harmonic analysis. In fact, this work is a prelude to the canonical representations of $\mathrm{Sp}(1,n)$ associated with representations of $\mathrm{Sp}(1)$ (cf. [DP]). In a special case, these representations have been studied by van Dijk and Hille [DH], but the introduction of canonical representations goes back to Berezin and to Gel′fand, Graev and Vershik. The main task is their decomposition into irreducible components, and, for this purpose, one needs the harmonic analysis we have developed in this paper.

The methods we employ take their roots in the work of Harish-Chandra and Godement, but their particular application we consider appears to be new. We have tried to keep a down-to-earth exposition in order to make the deep work of these authors accessible to a large mathematical audience. We have adopted Godement's prospective of trace spherical functions, but other points of view are possible (see [Dij]). Partial results have been previously obtained by Takahashi (but neither the Plancherel formula, nor the list of the positive definite spherical functions) and by Ørsted and Zhang (only – incomplete – results on the Plancherel formula). The Plancherel formula has been recently determined by Camporesi [Cam] in a much wider context than ours. His formula is however of very little use for practical purposes, and it does not even transparently show the possible splitting of the spectrum into continuous and discrete parts. Moreover, it has required the full Plancherel theorem on G, a tool by far more complicated than those employed for the known Plancherel theorem for the K-bi-invariant functions.

Let us now describe in more detail the background of the paper.

In [Go], Godement developed a general theory for the functions on a locally compact group G which are spherical with respect to a compact subgroup K. In his definition, the spherical functions on G arise from K-finite irreducible Banach representations of G. Let $g \mapsto T(g)$ be such a representation of G on $\mathcal{H}$. Suppose τ is an equivalence class of irreducible unitary representations of K that occurs in the restriction $T|_K$ of T to K. Let d_τ and χ_τ respectively denote the dimension and the character of τ. Set $\xi_\tau(k) = d_\tau \chi_\tau(k^{-1})$ for $k \in K$, and form the projection $E(\tau) = T|_K(\xi_\tau)$ of $\mathcal{H}$ onto the K-isotypic subspace of $\mathcal{H}$ of type τ. Then the spherical trace function $\zeta_{\tau,T}$ of type τ (shortly, τ-spherical function) on G associated with T is defined as the trace

$$\zeta_{\tau,T}(g) = \frac{1}{d_\tau}\ \mathrm{tr}[E(\tau)T(g)E(\tau)]\,.$$

$\zeta_{\tau,T}$ is said to be of height p when τ occurs p times ($p \geq 1$) in $T|_K$.

Suppose, as in our situation, that G is a semisimple Lie group. Then the τ-spherical functions of height 1 have much more manageable descriptions, either as common eigenfunctions of the left-invariant differential operators on G which are $\mathrm{Ad}(K)$-invariant, or as characters of the convolution algebra $\mathcal{D}(G;\chi_\tau)$ of all C^∞ compactly supported K-central functions on G with fixed K-type τ. Moreover, as proved by Godement, all τ-spherical functions are of height 1 when $\mathcal{D}(G;\chi_\tau)$ is commutative. In this case, the τ-spherical functions are the building blocks for the harmonic analysis on L^2-sections of the homogeneous vector bundle on G/K associated with τ. For example, the algebra $\mathcal{D}(G;\chi_\tau)$ is always commutative when G has finite center, K is maximally compact in G and τ is the trivial representation $\mathbf{1}$. Then the spherical functions of type $\mathbf{1}$ agree with the usual K-bi-invariant spherical functions on G. A less classical example of commutativity of $\mathcal{D}(G;\chi_\tau)$ is provided by Hermitian symmetric pairs (G,K) and 1-dimensional representations of K (cf. [Shi]).

Takahashi recognized in [T2] that if $G=\mathrm{Sp}(1,n)$ and $K=\mathrm{Sp}(1)\times\mathrm{Sp}(n)$, then the algebra $\mathcal{D}(G;\chi_\tau)$ is commutative for every irreducible representation τ of K which is trivial on $\mathrm{Sp}(n)$. He also explicitly computed some characters of $\mathcal{D}(G;\chi_\tau)$ (and it turns out that they are all !) in terms of Jacobi functions. The case $n=1$ has been previously studied by the same author in [T1].

Our paper is organized as follows. In Section 1 we recall some structural properties of $\mathrm{Sp}(1,n)$. Section 2 describes the commutativity of the algebra $\mathcal{D}(G;\chi_\tau)$ associated with a representation τ of K which is trivial on $Sp(n)$. Section 3 introduces the τ-spherical functions as characters of $\mathcal{D}(G;\chi_\tau)$ and gives their first properties. In Section 4 we find the differential equations satisfied by the spherical functions. This either provides us with their explicit expression in terms of Jacobi functions, or allows us to conclude that they are indeed all the spherical functions for $\mathrm{Sp}(1,n)$ associated with the given representation of $\mathrm{Sp}(1)\subset K$.

In Section 5 we write the spherical functions of type τ as the trace of the projection on the K-type τ of certain degenerate principal series representations of $\mathrm{Sp}(1,n)$ which have been studied by Howe and Tan [HT]. From this we can establish which among our spherical functions are positive definite. We underline the occurrence of a rather peculiar phenomenon: for a fixed representation τ, there are positive definite spherical functions arising from the complementary series of $\mathrm{Sp}(1,n)$ if and only if there are no positive definite spherical functions arising from the discrete series.

In Section 6 we prove that the τ-Abel transform is an isomorphism of $\mathcal{D}(G;\chi_\tau)$ onto the convolution algebra $\mathcal{D}_+(\mathbf{R})$ of the even C^∞ compactly supported functions on $\mathbf{R}$. The inversion formula is explicitly written. The Paley-Wiener Theorem for the τ-spherical transform is an immediate consequence. The final Section 7 contains the inversion formula and the Plancherel Theorem for the τ-spherical transform.

Similar results for $\mathrm{SU}(n,1)$ have been obtained as a specialization of the Hermitian symmetric case by Shimeno [Shi] and Heckman [HS, Part 1].

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1. THE FINE STRUCTURE OF $\mathrm{Sp}(1,n)$

Let $\mathbf{H}$ be the skew-field of the quaternions. Consider on the right $\mathbf{H}$-vector space $\mathbf{H}^{n+1}$ the Hermitian form

$$[x,y]=\bar{y}_0x_0-\bar{y}_1x_1-\cdots-\bar{y}_nx_n\,, \tag{1.1}$$

the bar sign denoting quaternionic conjugation: if $1,i,j,k$ are the quaternionic units and $q=a+ib+jc+kd\in\mathbf{H}$ (with $a,b,c,d\in\mathbf{R}$), then $\bar{q}=a-ib-jc-kd$. Let $G=\mathrm{Sp}(1,n)$ be the group $\mathrm{U}(1,n;\mathbf{H})$ of $(n+1)\times(n+1)$ matrices with coefficients in $\mathbf{H}$ which preserve this form. For $n=1$, G is called the De Sitter group. Let $\mathrm{Sp}(m)$ indicate the group $\mathrm{U}(m;\mathbf{H})$ of $m\times m$ matrices with coefficients in $\mathbf{H}$ which preserve the inner product $(x,y)=\bar{y}_1x_1+\cdots+\bar{y}_mx_m$ of $\mathbf{H}^m$. In particular, $\mathrm{Sp}(1)$ consists of the quaternions $q=a+i\,b+j\,c+k\,d$ with norm $|q|=\sqrt{a^2+b^2+c^2+d^2}$ equal to 1. $\mathrm{Sp}(1)$ is canonically isomorphic to $\mathrm{SU}(2)$. The group G acts on the projective space $P_n(\mathbf{H})$. Let Ω denote the image of the open set $\{x\in\mathbf{H}^{n+1}:[x,x]>0\}$ under the canonical map $\mathbf{H}^{n+1}\setminus\{0\}\to P_n(\mathbf{H})$. Then G acts transitively on Ω, and the stabilizer of the quaternionic line generated by the vector $(1,0,\dots,0)$ is the group

$$K=\left\{\begin{bmatrix} u & 0\\ 0 & U\end{bmatrix} : u\in\mathrm{Sp}(1),\,U\in\mathrm{Sp}(n)\right\}\equiv\mathrm{Sp}(1)\times\mathrm{Sp}(n)\,.$$

The homogeneous space G/K is called the hyperbolic quaternionic space. K is a maximally compact subgroup of G. G is connected and simply connected.

To study the fine structure of G, we consider its Lie algebra $\mathfrak{g}=\mathfrak{sp}(1,n)$. Let J be the $(n+1)\times(n+1)$ matrix $\operatorname{diag}(-1,1,\dots,1)$. For any matrix X of type $(n+1,n+1)$ with coefficients in $\mathbf{H}$ we set $X^*=J\overline{X}^tJ$, the symbol ^{t} denoting transposition.

The Lie algebra $\mathfrak{g}$ consists of the matrices X which verify the relation

$$X+X^*=0.$$

These are the matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ \overline{Z}_2^t & Z_3 \end{bmatrix}$$

with Z_1 and Z_3 anti-Hermitian of type $(1,1)$ and (n,n), respectively, and Z_2 arbitrary. Let θ be the anti-involutive automorphism of $\mathfrak{g}$ defined by

$$\theta X=JXJ.$$

Then θ is a Cartan involution with the usual decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Here $\mathfrak{k}$ is the Lie algebra of K. Let L be the following element of $\mathfrak{g}$:

$$L=\begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $L\in\mathfrak{p}$ and $\mathfrak{a}=\mathbf{R}L$ is a maximal Abelian subspace of $\mathfrak{p}$. We are going to diagonalize $\operatorname{ad}L$. The centralizer of L in $\mathfrak{k}$ is the subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ of the matrices

$$\begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with $u\in\mathbf{H}$, $u+\bar{u}=0$ and V a matrix of type $(n-1,n-1)$ satisfying $V+\overline{V}^t=\mathbf{0}$. The non-zero eigenvalues of $\operatorname{ad}L$ are $\alpha=1$, $-\alpha$, $\pm2\alpha$. The space $\mathfrak{g}_\alpha$ consists of the matrices

$$X=\begin{bmatrix} 0 & z^* & 0 \\ z & \mathbf{0} & -z \\ 0 & z^* & 0 \end{bmatrix}$$

where z is a matrix of type $(n-1,1)$ with coefficients in $\mathbf{H}$, and $z^*:=\bar{z}^t$. The real dimension of $\mathfrak{g}_\alpha$ is $m_\alpha=4(n-1)$. The space $\mathfrak{g}_{2\alpha}$ consists of the matrices of the form

$$X=\begin{bmatrix} w & 0 & -w \\ 0 & \mathbf{0} & 0 \\ w & 0 & -w \end{bmatrix}$$

with $w \in \mathbf{H}$, $w + \overline{w} = 0$. The dimension of $\mathfrak{g}_{2\alpha}$ is equal to $m_{2\alpha} = 3$. We have $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$.

Let A be the subgroup $\exp \mathfrak{a}$. This is the subgroup of the matrices

$$a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}$$

where t is a real number. The centralizer of A in K is the subgroup M of the matrices

$$m(u, V) = \begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with $u \in \mathrm{Sp}(1)$ and $V \in \mathrm{Sp}(n-1)$. The Lie algebra of M is $\mathfrak{m}$. The subspace $\mathfrak{n} = \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$ is a (real) nilpotent subalgebra. Set $N = \exp \mathfrak{n}$. This is the subgroup of the matrices

$$n(w, z) = \begin{bmatrix} 1 + w - \frac{1}{2}[z, z] & z^* & -w + \frac{1}{2}[z, z] \\ z & I & -z \\ w - \frac{1}{2}[z, z] & z^* & 1 - w + \frac{1}{2}[z, z] \end{bmatrix}$$

where $w \in \mathbf{H}$ satisfies $w + \overline{w} = 0$ and $z = [z_1, \dots, z_{n-1}]^t$ is a matrix of type $(n-1, 1)$ with coefficients in $\mathbf{H}$. We have set $z^* = \bar{z}^t$ and $[z, z] = -\bar{z}_1 z_1 - \cdots - \bar{z}_{n-1} z_{n-1}$.

The composition law in N is the following:

$$n(w, z) \cdot n(w', z') = n(w + w' + \Im[z, z'], z + z'),$$

where $\Im q := \frac{q - \bar{q}}{2}$ for $q \in \mathbf{H}$. The subgroups A and M normalize N:

$$a_t n(w, z) a_{-t} = n(e^{2t} w, e^t z),$$
$$m(u, V) n(w, z) m(u, V)^{-1} = n(u w \bar{u}, V z \bar{u}).$$

Let 2ρ be the trace of the restriction of $\mathrm{ad}\, L$ to $\mathfrak{n}$:

$$\rho = \frac{1}{2}(m_{\alpha} + 2m_{2\alpha}) = 2n + 1. \tag{1.2}$$

We have the Iwasawa decomposition $G = KAN = KNA$ and the corresponding integral formulas:

$$\int_G f(g)\, dg = \int_K \int_{-\infty}^{+\infty} \int_N f(k a_t n) e^{2\rho t}\, dk\, dt\, dn \tag{1.3}$$

$$= \int_K \int_N \int_{-\infty}^{+\infty} f(k n a_t)\, dk\, dn\, dt \tag{1.4}$$

for $f \in C_c(G)$. We adopt the usual notation $C_c(G)$ for the space of continuous functions on G with compact support. In the above formulas, $dn = dw\,dz$ ($n = n(w,z)$) and dk is the normalized Haar measure on K.

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \qquad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every $g \in G$ can be written as $g = k_1 k_2 a_t k_2'$ for uniquely determined $k_1 \in K_1$, $t \geq 0$ and for some $k_2, k_2' \in K_2$. Writing $g = [g_{ij}]_{i,j=0}^n$, we have

$$k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|\,. \tag{1.5}$$

If $g \notin K$, then $t > 0$ and k_2, k_2' are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After dg is normalized according to (1.3), the corresponding integral formula is

$$\int_G f(g)\,dg = \frac{1}{2}\left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1}\int_{K_2}\int_0^\infty\int_{K_2} f(k_1 k_2 a_t k_2')\Delta(t)\,dk_1\,dk_2\,dt\,dk_2' \tag{1.6}$$

where

$$\Delta(t) := 2^{2\rho}(\sinh t)^{4n-1}(\cosh t)^3\,. \tag{1.7}$$

2. The convolution algebra $\mathcal{D}(G;\chi_l)$

Let $\mathbf{N}/2$ be the set of nonnegative half-integers $\{0, 1/2, 1, 3/2, \dots\}$. Since $K_1 \equiv \mathrm{Sp}(1)$ is isomorphic to $\mathrm{SU}(2)$, $\mathbf{N}/2$ parametrizes the set of equivalence classes of unitary irreducible representations of K_1. We denote with the same symbol τ_l either the equivalence class corresponding to the parameter l or a fixed representative for it. Thus τ_l is a unitary irreducible representation of K_1 in a Hilbert space V_l of dimension $d_l = 2l+1$. We extend τ_l to a representation of K by setting $\tau_l \equiv \mathbf{1}$ on K_2. Each τ_l is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character $\chi_l = \operatorname{tr}\tau_l$ of τ_l satisfies $\chi_l(k^{-1}) = \chi_l(k)$, $k \in K$.

We denote by $\mathcal{D}(G;\tau_l)$ the convolution algebra of the compactly supported C^∞ maps $F\colon G \to \mathrm{End}(V_l)$ satisfying

$$F(kxk') = \tau_l(k)F(x)\tau_l(k') \qquad (k, k' \in K, x \in G). \tag{2.8}$$

Let $\mathcal{D}(G)$ be the convolution algebra of the C^∞ compactly supported complex valued functions on G. Then $\mathcal{D}(G;\tau_l)$ is isomorphic to the subalgebra $\mathcal{D}(G;\chi_l)$ of $\mathcal{D}(G)$ consisting of the functions $f \in \mathcal{D}(G)$ satisfying

$$f^0 = f \tag{2.9}$$

and

$$f * d_l\chi_l = f, \tag{2.10}$$

where

$$f^0(x) := \int_K f(kxk^{-1})\, dk. \tag{2.11}$$

The isomorphism is given by $F \longmapsto d_l \operatorname{tr} F$ (see e.g. [Dij], Theorem 1.1).

The commutativity of the algebra $\mathcal{D}(G;\chi_l)$ can be deduced from the fact that the restriction $\tau_l|_M$ of τ_l to M is multiplicity free (according to the general criterion by Deitmar, cf. [Dei] Theorem 3, the commutativity of $\mathcal{D}(G;\chi_l)$ and the multiplicity freeness of $\tau_l|_M$ are in fact equivalent). An elementary direct argument by Takahashi proves the commutativity of a convolution algebra which is slightly bigger than $\mathcal{D}(G;\chi_l)$. Let $\mathcal{D}_1(G)$ be the subalgebra of $\mathcal{D}(G)$ consisting of the functions $f \in \mathcal{D}(G)$ satisfying

$$f(k_2k_1gk_1^{-1}k_2') = f(g), \qquad g \in G, k_1 \in K_1, k_2 \in K_2. \tag{2.12}$$

Clearly $\mathcal{D}(G;\chi_l) \subset \mathcal{D}_1(G)$. Moreover $\mathcal{D}_1(G) = \oplus_l \mathcal{D}(G;\chi_l)$. Showing that

$$f(g^{-1}) = f(g) \qquad \text{for all } f \in \mathcal{D}_1(G) \text{ and } g \in G, \tag{2.13}$$

one proves the following lemma.

2.1. LEMMA ([T2], Proposition 1). *The algebra $\mathcal{D}_1(G)$ is commutative.*

2.2. LEMMA (cf. [T2], Lemma 2). *For every function $f \in \mathcal{D}(G;\chi_l)$,*

$$f(g) = f(k_1a_t) = \frac{1}{d_l}\chi_l(k_1)f(a_t) \tag{2.14}$$

if $g = k_1k_2a_tk_2'$ $(k_1 \in K_1; k_2, k_2' \in K_2;\ t \in \mathbf{R})$.

Proof. For $F \in \mathcal{D}(G;\tau_l)$, $F(a_t)$ is a scalar multiple of the identity, since it commutes with $\tau_l(k_1)$ for all $k_1 \in K_1$. □

2.3. REMARK. Formula (2.13) and Lemma 2.2 remain true for all continuous functions f such that $f = f^0$, $f * d_l\chi_l = f$.

3. THE τ_l-SPHERICAL FUNCTIONS

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}=\mathfrak{sp}(1,n)$. The elements of $\mathfrak{U}(\mathfrak{g})$ are considered as left-invariant differential operators on $G=\mathrm{Sp}(1,n)$ acting on C^∞ functions on the right:

$$f(g;X) := \frac{d}{dt}f(g\exp tX)\Big|_{t=0} \qquad (f\in C^\infty(G),\ X\in\mathfrak{g},\ g\in G).$$

We adopt Harish-Chandra's notation $f(g;D)$ for the image of $f\in C^\infty(G)$ under the right action of $D\in\mathfrak{U}(\mathfrak{g})$. The set of K-invariant elements of $\mathfrak{U}(\mathfrak{g})$ is denoted by $\mathfrak{U}(\mathfrak{g})^K$.

3.1. THEOREM ([Go], Theorems 8, 10 and 14; [GaV], Theorems 1.3.14, 1.4.5 and Proposition 1.4.4). *Let $l\in\mathbf{N}/2$ be fixed. Let ζ be a complex-valued continuous function on G satisfying (2.9), (2.10) and $\zeta(e)=1$ (e is the unit element of G). The following statements are mutually equivalent.*

1. *The mapping $f\mapsto\int_G f(g)\zeta(g)\,dg$ is an algebra homomorphism of $\mathcal{D}(G;\chi_l)$ into $\mathbf{C}$.*
2. *ζ satisfies the functional equation*

$$\int_K \zeta(kg_1k^{-1}g_2)\,dk=\zeta(g_1)\zeta(g_2) \tag{3.15}$$

   for all $g_1,g_2\in G$.
3. *ζ is a common eigenfunction of the elements of $\mathfrak{U}(\mathfrak{g})^K$.*

A function ζ satisfying the equivalent conditions of Theorem 3.1 is called a *spherical function of type τ_l (and height 1)* or briefly a *τ_l-spherical function* [1]).

Observe that Condition 3 implies in particular that every τ_l-spherical function is analytic on G because $\mathfrak{U}(\mathfrak{g})^K$ contains an elliptic differential operator.

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[1]) Our definition of τ_l-spherical function ζ is formally less restrictive than Godement's, which also requires ζ to be quasi-bounded with respect to some seminorm on G (cf. [Go], p. 519). It is possible to show (e.g. [GaV], Theorem 1.3.14) that each of the conditions for ζ given by Theorem 3.1 is equivalent to the existence of an irreducible Fréchet representation $(T,\mathcal{H})$ – with a d_l-dimensional K-isotypic subspace $\mathcal{H}(\tau_l)$ of type τ_l – for which

$$\zeta=\frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)TE(\tau_l)]\,, \tag{$*$}$$

$E(\tau_l)$ being the projection of $\mathcal{H}$ onto $\mathcal{H}(\tau_l)$. $(T,\mathcal{H})$ can be chosen to be a Banach representation of G if and only if ζ is quasi-bounded with respect to some seminorm on G. In Section 5 we will determine, for each τ_l-spherical function ζ, an irreducible *Hilbert* representation $(T,\mathcal{H})$ for which $(*)$ holds. It follows, in particular, that the condition of quasi-boundedness is, in our case, automatically satisfied.

For complex-valued functions f on G and F on $\mathbf{R}$, we set

$$f^*(g) = \overline{f(g^{-1})} \qquad (g \in G)$$
$$F^*(t) = \overline{F(-t)} \qquad (t \in \mathbf{R})\,.$$

The *τ_l-Abel transform* of $f \in \mathcal{D}(G;\chi_l)$ is the function $\mathcal{A}_l f$ on $\mathbf{R}$ defined by

$$\mathcal{A}_l f(t) = \frac{1}{d_l^2}\, e^{\rho t} \int_N f(a_t n)\, dn\,. \tag{3.16}$$

Its properties are summarized in the following proposition.

3.2. PROPOSITION. *For all $f \in \mathcal{D}(G;\chi_l)$, $\mathcal{A}_l f$ is a C^∞ function on $\mathbf{R}$ with compact support. If $f, f_1, f_2 \in \mathcal{D}(G;\chi_l)$ and $a_1, a_2 \in \mathbf{C}$, then*

$$(\mathcal{A}_l f)^* = \mathcal{A}_l(f^*)\,, \tag{3.17}$$

$$\mathcal{A}_l(a_1 f_1 + a_2 f_2) = a_1 \mathcal{A}_l f_1 + a_2 \mathcal{A}_l f_2\,, \tag{3.18}$$

$$\mathcal{A}_l(f_1 * f_2) = \mathcal{A}_l f_1 * \mathcal{A}_l f_2\,. \tag{3.19}$$

Formula (3.17) is equivalent to the fact that $\mathcal{A}_l f$ is an even function.

Proof. Formulas (3.17)–(3.19) are immediately proven by passing to $\mathcal{D}(G;\tau_l)$. For the last statement, recall that $f(g^{-1}) = f(g)$ for $f \in \mathcal{D}(G;\chi_l)$. $\square$

The following lemma relates our definition of $\mathcal{A}_l$ to the definition often found in the literature (cf. e.g. [W2], p. 34).

3.3. LEMMA. *For $f \in \mathcal{D}(G;\chi_l)$ one has*

$$\int_N f(k a_t n)\, dn = \frac{1}{d_l}\chi_l(k) \int_N f(a_t n)\, dn\,.$$

Proof. Let $F \in \mathcal{D}(G;\tau_l)$. Then $\int_N F(a_t n)\, dn$ commutes with $\tau(m)$ $(m \in M)$ so with $\tau(k_1)$ $(k_1 \in K_1)$, hence is a scalar multiple of the identity. The lemma follows by taking traces. $\square$

We now use the τ_l-Abel transform to construct τ_l-spherical functions. Because of Proposition 3.2, for any complex number s, the map

$$\lambda_s : f \longmapsto \int_{-\infty}^{\infty} \mathcal{A}_l f(t) e^{-st}\, dt \tag{3.20}$$

is an algebra homomorphism of $\mathcal{D}(G;\chi_l)$ into $\mathbf{C}$.

Set

$$\alpha_{l,s}(ka_tn) = \frac{1}{d_l}\chi_l(k)e^{-(s+\rho)t}\,. \tag{3.21}$$

Since $f = f * d_l\chi_l$ and $\chi_l(k^{-1}) = \chi_l(k)$ for $k \in K$, for every $f \in \mathcal{D}(G;\chi_l)$

$$\begin{aligned}\lambda_s(f) &= \frac{1}{d_l}\int_K\int_{-\infty}^{\infty}\int_N f(ka_tn)\chi_l(k)e^{(-s+\rho)t}\,dk\,dt\,dn\\ &= \int_G f(g)\alpha_{l,s}(g)\,dg\\ &= \int_G f(g)\int_K \alpha_{l,s}(kgk^{-1})\,dk\,dg\\ &= \int_G f(g)\zeta_{l,s}(g)\,dg \qquad (3.22)\end{aligned}$$

with

$$\zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1})\,dk\,. \tag{3.23}$$

One easily checks that $\zeta_{l,s}$ satisfies $\zeta_{l,s} = \zeta_{l,s}{}^0$, $\zeta_{l,s}*d_l\chi_l = \zeta_{l,s}$ and $\zeta_{l,s}(e) = 1$. Thus $\zeta_{l,s}$ is a τ_l-spherical function. It will be shown in the next section that any τ_l-spherical function is of the form (3.24).

By Remark 2.3, we have

$$\zeta_{l,s}(g) = \frac{1}{d_l}\chi_l(k_1)\zeta_{l,s}(a_t) \quad \text{for } g = k_1k_2a_tk_2'\,, \tag{3.24}$$

so $\zeta_{l,s}$ is uniquely determined by its restriction to A.

4. THE DIFFERENTIAL EQUATION FOR THE τ_l-SPHERICAL FUNCTIONS

For a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$, let $\mathfrak{u}_{\mathbf{C}}$ denote the complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by $\mathfrak{u}$. The universal enveloping algebra $\mathfrak{U}(\mathfrak{u})$ of $\mathfrak{u}_{\mathbf{C}}$ is considered as a subalgebra of $\mathfrak{U}(\mathfrak{g})$.

The representation τ_l of K_1 induces differentiated representations of the Lie algebra $\mathfrak{k}_1$ of K_1 and of the universal enveloping algebra $\mathfrak{U}(\mathfrak{k}_1)$ of $(\mathfrak{k}_1)_{\mathbf{C}}$. We indicate these representations with the same letter τ_l. Let $\mathfrak{k}_2$ be the Lie algebra of K_2. Every element $Y \in \mathfrak{k}_{\mathbf{C}}$ can be uniquely decomposed as

$Y = Y^{(1)} + Y^{(2)}$ with $Y^{(j)} \in (\mathfrak{k}_j)_{\mathbf{C}}$, $j = 1, 2$. The symbol χ_l will also be used for the $\mathbf{C}$-linear map on $\mathfrak{U}(\mathfrak{k})$ defined by

$$\chi_l(Y_1 \cdots Y_m) := \operatorname{tr}\left[\tau_l(Y_1^{(1)}) \cdots \tau_l(Y_m^{(1)})\right]$$

for $Y_1, \ldots, Y_m \in \mathfrak{k}_{\mathbf{C}}$.

The Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ gives $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{k})\mathfrak{U}(\mathfrak{a})\mathfrak{U}(\mathfrak{n}) \cong \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a}) \oplus \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbf{C}}$. Let $P \colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a})$ be the corresponding projection. For $s \in \mathbf{C}$, let e_s be the $\mathbf{C}$-linear map on $\mathfrak{U}(\mathfrak{a})$ defined by

$$e_s(L^m) := (-1)^m (s + \rho)^m \qquad \text{for every integer } m \geq 0\,.$$

Define $p_{l,s} \colon \mathfrak{U}(\mathfrak{g}) \to \mathbf{C}$ to be the composition $p_{l,s} := \left(\frac{1}{d_l}\chi_l \otimes e_s\right) \circ P$, where as before $d_l = \dim \tau_l$.

4.1. PROPOSITION. *Let $\zeta_{l,s}$ be the function defined by Formula (3.23). For every $D \in \mathfrak{U}(\mathfrak{g})^K$ and $g \in G$*

$$\zeta_{l,s}(g; D) = p_{l,s}(D)\zeta_{l,s}(g)\,. \tag{4.25}$$

Proof. Because of Theorem 3.1, $\zeta_{l,s}$ is an eigenfunction of every $D \in \mathfrak{U}(\mathfrak{g})^K$. The eigenvalue corresponding to $D \in \mathfrak{U}(\mathfrak{g})^K$ is $\zeta_{l,s}(e; D)$ because $\zeta_{l,s}(e) = 1$. Since D is K-invariant, $\zeta_{l,s}(e; D) = \alpha_{l,s}(e; D)$. Write $D = \sum_i y_i x_i + \sum_j n_j$ with $y_i \in \mathfrak{U}(\mathfrak{k})$, $x_i \in \mathfrak{U}(\mathfrak{a})$ and $n_j \in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbf{C}}$. Then $\alpha_{l,s}(e; D) = \sum_i \alpha_{l,s}(e; y_i x_i)$ because $\alpha_{l,s}(gn) = \alpha_{l,s}(g)$ for $g \in G$ and $n \in N$. To compute $\alpha_{l,s}(e; y_i x_i)$, assume without loss of generality that $x_i = L^{m_i}$ and that $y_i = Y_1 \cdots Y_m$ with $Y_j \in \mathfrak{k}$. The definition of $\alpha_{l,s}$ gives

$$\alpha_{l,s}(e; y_i x_i) = \frac{1}{d_l}\chi_l(y_i)(-1)^m (s + \rho)^m = p_{l,s}(y_i x_i)\,.$$

Thus $\zeta_{l,s}(e; D) = p_{l,s}(D)$. $\square$

Let $\delta_l(D)$ denote the τ_l-radial component on $A^+ := \{a_t : t > 0\}$ of the differential operator $D \in \mathfrak{U}(\mathfrak{g})$; that is, the unique differential operator on A^+ satisfying

$$f(a_t; \delta_l(D)) = f(a_t; D)$$

for all $f \in \mathcal{D}(G; \chi_l)$ and $t > 0$. Proposition 4.1 immediately implies

4.2. COROLLARY. *$\zeta_{l,s}$ is an eigenfunction of the τ_l-radial component on A^+ of every K-invariant differential operator on G:*

$$\zeta_{l,s}(a_t; \delta_l(D)) = p_{l,s}(D)\zeta_{l,s}(a_t) \qquad (D \in \mathfrak{U}(\mathfrak{g})^K,\ t > 0)\,. \tag{4.26}$$

We now write (4.26) explicitly in the case D is the Casimir operator ω of $\mathfrak{g}$. Let B denote the Cartan-Killing form of $\mathfrak{g}_{\mathbf{C}}$ ($\cong \mathfrak{sp}(1+n,\mathbf{C})$). If $X, Y \in \mathfrak{sp}(1,n)$, then

$$B(X,Y) = 4(n+2)\,\Re\,\mathrm{tr}(XY)$$

where $\Re$ denotes the quaternionic real part: $\Re q = \frac{q+\bar{q}}{2}$ for $q \in \mathbf{H}$. The bilinear form $B_\theta(X,Y) := -B(X,\theta Y)$ is an inner product on $\mathfrak{g}$. Orthonormality will be considered with respect to B_θ.

Let $\{Z_j\}_{j=1}^m$ ($m := 2n^2+n$) and $\{X_{\beta,j}\}_{j=1}^{m_\beta}$ ($\beta \in \{\alpha, 2\alpha\}$) be orthonormal bases in $\mathfrak{m}$ and in $\mathfrak{g}_\beta$ respectively. Define $X_{-\beta,j} = -\theta(X_{\beta,j})$ for $\beta \in \{\alpha,2\alpha\}$ and $j = 1,\dots,m_\beta$. Then $\{X_{-\beta,j}\}_{j=1}^{m_\beta}$ is an orthonormal basis for $\mathfrak{g}_{-\beta}$, and $B(X_{\beta,i}, X_{-\beta,j}) = \delta_{ij}$. Moreover, for all $j = 1,\dots,m_\beta$, $H_\beta := [X_{\beta,j}, X_{-\beta,j}]$ is the unique element of $\mathfrak{a}$ satisfying $B(H_\beta, L) = \beta(L)$, i.e.

$$H_\beta = \frac{h_\beta}{8(n+2)}\,L \quad \text{with} \quad h_\beta = \begin{cases} 1 & \text{if } \beta = \alpha \\ 2 & \text{if } \beta = 2\alpha\,. \end{cases}$$

Set $H_1 := \frac{L}{\sqrt{8(n+2)}}$, a unit vector in $\mathfrak{a}$. Then, if $D_{\beta,j} := X_{\beta,j}X_{-\beta,j} + X_{-\beta,j}X_{\beta,j}$, we have (cf. [GaV], p. 132)

$$\begin{aligned}
(4.27) \qquad \omega &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta\in\{\alpha,2\alpha\}} \sum_{j=1}^{m_\beta} D_{\beta,j} \\
&= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta\in\{\alpha,2\alpha\}} m_\beta H_\beta + 2\sum_{\beta\in\{\alpha,2\alpha\}} \sum_{j=1}^{m_\beta} X_{\beta,j}X_{-\beta,j}
\end{aligned}$$

where

$$(4.28) \qquad \omega_{\mathfrak{m}} := -\sum_{j=1}^{m} Z_j^2\,.$$

Hence

$$P(\omega) = \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta\in\{\alpha,2\alpha\}} m_\beta H_\beta = \omega_{\mathfrak{m}} + \frac{L^2 + 2\rho L}{B(L,L)}\,,$$

from which we conclude

$$(4.29) \qquad p_{l,s}(\omega) = p_{l,s}(\omega_{\mathfrak{m}}) + \frac{(s+\rho)^2 - 2\rho(s+\rho)}{B(L,L)} = \frac{1}{d_l}\chi_l(\omega_{\mathfrak{m}}) + \frac{s^2-\rho^2}{8(n+2)}\,.$$

To compute $\delta_l(\omega)$ we use Formula (4.27). Observe first that if $f \in \mathcal{D}(G;\chi_l)$ and $Y \in \mathfrak{U}(\mathfrak{k})$, then $f(a_t;Y) = \frac{1}{d_l}\chi_l(Y)$. Hence $\delta_l(Y) = \frac{1}{d_l}\chi_l(Y)$. In particular,

$$\delta_l(\omega_{\mathfrak{m}}) = \frac{1}{d_l}\chi_l(\omega_{\mathfrak{m}})\,. \tag{4.30}$$

Write

$$X_{\beta,j} = Y_{\beta,j} + P_{\beta,j} \qquad \text{with} \qquad Y_{\beta,j} \in \mathfrak{k}, P_{\beta,j} \in \mathfrak{p}\,. \tag{4.31}$$

A standard computation (cf. e.g. [W2], p. 278) then gives for $f \in \mathcal{D}(G;\chi_l)$ and $t > 0$

$$f(a_t;D_{\beta,j}) = \coth(t\beta(L))f(a_t;H_\beta) + \frac{4}{d_l}\frac{1-\cosh(t\beta(L))}{\sinh^2(t\beta(L))}\,\chi_l({Y_{\beta,j}}^2)f(a_t)$$

i.e.

$$\delta_l(D_{\beta,j}) = \coth(t\beta(L))H_\beta + \frac{4}{d_l}\frac{1-\cosh(t\beta(L))}{\sinh^2(t\beta(L))}\,\chi_l({Y_{\beta,j}}^2)\,. \tag{4.32}$$

Notice that $\chi_l(Y_{\alpha,j}^2) = 0$ for all $j = 1, \dots, m_\alpha$.

For $h = i, j, k$, let Y_h denote the tangent vector at e to the 1-parameter subgroup $t \mapsto \cos t + h \sin t$ in $\mathrm{Sp}(1)$. Explicit choices of the orthonormal bases in $\mathfrak{m}$ and $\mathfrak{g}_{2\alpha}$ prove that

$$\chi_l(\omega_{\mathfrak{m}}) = -2\sum_{j=1}^{3}\chi_l(Y_{2\alpha,j}^2) = -\frac{1}{8(n+2)}\sum_{h\in\{i,j,k\}} \mathrm{tr}\left[\tau_l(Y_h)^2\right]\,. \tag{4.33}$$

As shown in [T1], p. 381, there exists an orthonormal basis $\{v_p\}_{p=-l}^{l}$ in V_l such that

$$\begin{aligned}
\tau_l(Y_i)v_p &= -2ipv_p\\
\tau_l(Y_j)v_p &= -i\alpha_{p+1}^l v_{p+1}\\
\tau_l(Y_k)v_p &= -\alpha_{p+1}^l v_{p+1} + \alpha_p^l v_{p-1}
\end{aligned}$$

where

$$\alpha_p^l := [(l+p)(l-p+1)]^{1/2}\,.$$

It follows that for $h = i, j, k$

$$\mathrm{tr}\left[\tau_l(Y_h)^2\right] = -\frac{4}{3}l(l+1)(2l+1)\,. \tag{4.34}$$

Identify A with $\mathbf{R}$ and L with $\frac{d}{dt}$ under the isomorphism $t \mapsto \exp(tL) = a_t$. Formulas (4.27), (4.30) and (4.32)–(4.34) then prove the following proposition.

4.3. PROPOSITION. *Let τ_l be an irreducible unitary representation of K_1 of dimension $2l+1$. Then*

1. *The τ_l-radial component of the Casimir operator ω is*

$$\delta_l(\omega) = \frac{1}{8(n+2)}\left\{\frac{d^2}{dt^2}+[(4n-1)\coth t+3\tanh t]\frac{d}{dt}+\frac{4l(l+1)}{\cosh^2 t}+4l(l+1)\right\}.$$

2. *For every $s\in\mathbf{C}$*

$$p_{l,s}(\omega) = \frac{1}{8(n+2)}[4l(l+1)+s^2-\rho^2]. \tag{4.35}$$

3. *For every $s\in\mathbf{C}$, the function $\zeta_{l,s}(t) := \zeta_{l,s}(a_t)$ satisfies the differential equation on $(0,+\infty)$*

$$u''+[(4n-1)\coth t+3\tanh t]u'+\frac{4l(l+1)}{\cosh^2 t}u = (s^2-\rho^2)\,u. \tag{4.36}$$

The substitution $v(t)=(\cosh t)^{-2l}u(t)$ transforms the differential equation (4.36) into the Jacobi differential equation (cf. [K2], p. 6)

$$v''+[(4n-1)\coth t+(4l+3)\tanh t]v' = (s^2-\tilde{\rho}^2)v \tag{4.37}$$

with parameters $\alpha=2n-1$ and $\beta=2l+1$. Here $\tilde{\rho}:=\alpha+\beta+1=\rho+2l$. The Jacobi function

$$\begin{aligned}\phi_{is}^{(2n-1,2l+1)}(t) :=& F\left(\frac{\tilde{\rho}+s}{2},\frac{\tilde{\rho}-s}{2};2n;-\sinh^2 t\right)\\ =& F\left(\frac{\rho+s}{2}+l,\frac{\rho-s}{2}+l;2n;-\sinh^2 t\right)\end{aligned} \tag{4.38}$$

is the unique solution v to (4.37) satisfying $v(0)=1$, $v'(0)=0$. (In (4.38), $F(a,b;c;z)$ denotes the analytic branch on $\mathbf{C}\setminus[1,\infty)$ of the hypergeometric function.)

The τ_l-spherical function $\zeta_{l,s}$ is therefore explicitly given by

$$\begin{aligned}\zeta_{l,s}(t) := \zeta_{l,s}(a_t) &= (\cosh t)^{2l}\,\phi_{is}^{(2n-1,2l+1)}(t)\\ &= (\cosh t)^{2l}F\left(\frac{\rho+s}{2}+l,\frac{\rho-s}{2}+l;2n;-\sinh^2 t\right).\end{aligned} \tag{4.39}$$

Formula (4.39) has been previously determined by Takahaski ([T2], Formula (7), p. 225) by direct integration of (3.23), using the following expression of χ_l in terms of Gegenbauer polynomials:

$$\chi_l(k_1) = C_{2l}^1(\Re k_1) = \frac{\sin((2l+1)\vartheta)}{\sin\vartheta} \qquad \text{if} \quad \Re k_1=\cos\vartheta. \tag{4.40}$$

Formula (4.35) shows that $p_{l,s}(\omega)$ is an even function of s which assumes arbitrary complex values as s varies in $\mathbf{C}$. The following corollary can therefore be deduced from Theorem 3.1 and Proposition 4.3.

4.4. COROLLARY. *The τ_l-spherical functions are exactly the functions $\{\zeta_{l,s} : s \in \mathbf{C}\}$ given by Formulas (3.24) and (4.39). Further, $\zeta_{l,s}$ satisfies $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$ for all $g \in G$. Moreover, $\zeta_{l,s} = \zeta_{l,s'}$ if and only if $s = \pm s'$.*

The functional equation (3.15) with $g_1 = a_t$ and $g_2 = a_\tau$ becomes (cf. [T2], Théorème 1, p. 227)

$$\zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t,\tau,u)\zeta_{l,s}(u)\Delta(u)\,du \tag{4.41}$$

where Δ is as in (1.7) and the kernel $K_l(t,\tau,u)$ is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2\cosh t \cosh \tau \cosh u}\,.$$

Then

$$\begin{aligned} K_l(t,\tau,u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n-\frac{1}{2})} \frac{(\cosh t \cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}} (1-B^2)^{2n-\frac{3}{2}} \\ \times F\Big(2n+2l, 2n-2l-2; 2n-\frac{1}{2}; \frac{1}{2}(1-B)\Big) \end{aligned} \tag{4.42}$$

if $B < 1$, and $K_l(t,\tau,u) := 0$ if $B \geq 1$. Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all $l \in \mathbf{R}$ satisfying $2n-1 > 2l \geq 0$.

5. THE POSITIVE DEFINITE τ_l-SPHERICAL FUNCTIONS

A continuous function ζ on a locally compact group G is said to be *positive definite* if for every $f \in C_c(G)$

$$\int_G\int_G \zeta(x^{-1}y)f(x)\overline{f(y)}\,dx\,dy \geq 0\,.$$

In this section we establish which among the $\zeta_{l,s}$ are positive definite.

Let us first introduce some notation and recall some definitions. Let G be a semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. $\mathfrak{g}$ and $\mathfrak{k}$ ($\subset \mathfrak{g}$) are the Lie algebras of G and K, respectively. A (strongly continuous) representation T of G on a Banach space $\mathcal{H}$ is denoted by $(T,\mathcal{H})$. We may simply speak of the representation T if $\mathcal{H}$ is understood. Irreducibility for T always means topological irreducibility (= no closed proper invariant subspaces). Let $\widehat{K}$ denote the set of equivalence classes of finite dimensional irreducible representations of K. We say that $\tau \in \widehat{K}$ occurs in $T|_K$ if there exists a finite dimensional $T|_K$-invariant subspace V of $\mathcal{H}$ so that $(T|_K, V) \in \tau$. The linear span of all these subspaces V is the K-isotypic subspace of $\mathcal{H}$ of type τ, denoted $\mathcal{H}(\tau)$. If d_τ is the dimension of τ and χ_τ is its character, then

$$E_T(\tau) = d_\tau \int_K T(k^{-1})\chi_\tau(k)\, dk$$

is a continuous projection of $\mathcal{H}$ onto $\mathcal{H}(\tau)$. We set $\mathcal{H}_K = \sum_{\tau\in K}\mathcal{H}(\tau)$. T is said to be K-finite if $\dim\mathcal{H}(\tau) < \infty$ for all $\tau \in \widehat{K}$. A Hilbert representation $(T, \mathcal{H})$ is said to be admissible if it is K-finite and if $T|_K$ acts on $\mathcal{H}$ by unitary operators.

A representation U of an (associative or Lie) algebra $\mathcal{A}$ on a $\mathbf{C}$-vector space E is denoted (U, E). The term $\mathcal{A}$-module is also used. Irreducibility for U always means algebraic irreducibility (= no proper invariant subspaces). Let $\widehat{\mathfrak{k}_\mathbf{C}}$ denote the set of equivalence classes of finite dimensional simple $\mathfrak{k}_\mathbf{C}$-modules. The sum of all simple $\mathfrak{k}_\mathbf{C}$-submodules of E which are in the class $\delta \in \widehat{\mathfrak{k}_\mathbf{C}}$ is denoted by $E(\delta)$. (U, E) is said $\mathfrak{k}$-finite if $\dim E(\delta) < \infty$ for all $\delta \in \widehat{\mathfrak{k}_\mathbf{C}}$ and if $E = \sum_{\delta\in\widehat{\mathfrak{k}_\mathbf{C}}} E(\delta)$.

Every K-finite irreducible representation $(T, \mathcal{H})$ of G induces a $\mathfrak{k}$-finite irreducible representation $(T_K, \mathcal{H}_K)$ of $\mathfrak{U}(\mathfrak{g})$ by differentiation. If, moreover, $\mathcal{H}$ is Hilbert and T is unitary, then $\mathfrak{g}$ acts on $\mathcal{H}_K$ by skew-adjoint operators: $\langle T_K(X)\varphi, \psi\rangle = -\langle\varphi, T_K(X)\psi\rangle$ for all $X \in \mathfrak{g}$ and all $\varphi, \psi \in \mathcal{H}_K$. Two K-finite representations $(T, \mathcal{H})$, $(T', \mathcal{H}')$ of G are said to be *infinitesimally equivalent* if the representations $(T_K, \mathcal{H}_K)$, $(T'_K, \mathcal{H}'_K)$ of $\mathfrak{U}(\mathfrak{g})$ are equivalent.

Assume G is simply connected (which is the case for $G = \mathrm{Sp}(1,n)$). It is a result of Harish-Chandra ([HC1], Theorem 9; see also [W1], pp. 330–331) that if (U, S) is an algebraically irreducible $\mathfrak{k}$-finite representation of $\mathfrak{U}(\mathfrak{g})$ and if S can be endowed with a positive definite Hermitian form $\langle\cdot,\cdot\rangle$ for which $\mathfrak{g}$ acts on $(S, \langle\cdot,\cdot\rangle)$ via skew-adjoint operators, then there is a unique unitary irreducible representation $\widetilde{T}$ of G on the Hilbert completion $\widetilde{\mathcal{H}}$ of S with respect to $\langle\cdot,\cdot\rangle$ so that $\widetilde{\mathcal{H}}_K = S$ and $\widetilde{T}_K = U$. We say in this case that (U, S) – or simply S if U is understood – is *unitarizable*. If, in particular, $(U, S) = (T_K, \mathcal{H}_K)$ for a K-finite irreducible representation $(T, \mathcal{H})$ of G, then $(T, \mathcal{H})$ and $(\widetilde{T}, \widetilde{\mathcal{H}})$ are infinitesimally equivalent. The converse is also obvious: if $(T, \mathcal{H})$ is an irreducible K-finite representation of G which is infinitesimally equivalent to a unitary Hilbert representation $(\widetilde{T}, \widetilde{\mathcal{H}})$ of G, then $(T_K, \mathcal{H}_K)$ is unitarizable.

As we are going to show, the τ_l-spherical functions can be written as

$$\zeta_{l,s}(g) = \frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)T_{l,s}(g)E(\tau_l)] = \frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)T_{l,s}(g)]$$

for certain admissible irreducible Hilbert representations $(T_{l,s}, \mathcal{H}_{l,s})$ of $G = \mathrm{Sp}(1,n)$ satisfying $\dim \mathcal{H}_{l,s}(\tau_l) = d_l$ (for the second equality see e.g. [HC2], Lemma 1). The positive definite $\zeta_{l,s}$ can then be selected by applying the following theorem.

5.1. THEOREM ([Sak], Theorem 3; [B], I.4.8, p. 44). *$\zeta_{l,s}$ is positive definite if and only if $(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation.*

Realize τ_l as a unitary representation on a $(2l+1)$-dimensional Hilbert space V_l with inner product $\langle \cdot, \cdot \rangle_l$. For all $s \in \mathbf{C}$, define a representation $\theta_{l,s}$ of $P = MAN$ on V_l by

$$\theta_{l,s}(ma_t n) = e^{-(s-\rho)t}\tau_l(m)\,.$$

Consider the representation $T'_{l,s} = \mathrm{Ind}_P^G(\theta_{l,s})$ of $G = \mathrm{Sp}(1,n)$: the representation space is the Hilbert completion $\mathcal{H}'_{l,s}$ of the set of the C^∞ functions $F\colon G \to V_l$ satisfying

$$F(gp) = \theta_{l,s}(p^{-1})F(g) = e^{(s-\rho)t}\tau_l(m^{-1})F(g)\,, \qquad g \in G,\ p = ma_t n \in P\,,$$

with respect to the inner product

$$(F_1, F_2)_l = \int_K \langle F_1(k), F_2(k)\rangle_l\, dk\,.$$

G acts according to

$$(T'_{l,s}(g)F)(g') = F(g^{-1}g'), \qquad g, g' \in G\,.$$

$T'_{l,s}$ is admissible, but need not be irreducible.

The following lemma is a straightforward generalization of the result in Section 16, pp. 526–528, of [Go]. We therefore omit its proof.

5.2. LEMMA. *For all $l \in \mathbf{N}/2$ and $s \in \mathbf{C}$, let $E'(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto its K-isotypic subspace of type τ_l. Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l}\,\mathrm{tr}[E'(\tau_l)T'_{l,s}(g)]\,.$$

The composition series structure and unitarity for the $T'_{l,s}$ have been determined by Howe and Tan with infinitesimal methods. In [HT], the results about the $T'_{l,s}$ are deduced from those obtained for a certain family of representations of $\mathrm{Sp}(1,n)\times\mathbf{H}^\times$ which are equivalent to $T'_{l,s}\otimes\tau_{l,s}$. Here $\mathbf{H}^\times=\mathbf{R}_+^\times\cdot\mathrm{Sp}(1)$ denotes the group of quaternionic dilations, acting on the space V_l of τ_l according to

$$\tau_{l,s}(h)=|h|^{s-\rho}\tau_l\left(h/|h|\right), \qquad h\in\mathbf{H}^\times.$$

5.3. THEOREM ([HT], Theorem 5.6 and p. 58).

1. $(\mathcal{H}'_{l,s})_K$ *is equivalent as a* $\mathfrak{U}(\mathfrak{g})$*-module to* $(\mathcal{H}'_{l,-s})_K$.
2. $(\mathcal{H}'_{l,s})_K$ *is a reducible* $\mathfrak{U}(\mathfrak{g})$*-module if and only if* $s\in\mathbf{Z}$, $s\equiv 2(l-n)+1 \pmod 2$ *and* $s\notin(2l-\rho+2,-2l+\rho-2)$.
3. *Suppose* $(\mathcal{H}'_{l,s})_K$ *irreducible. Then* $(\mathcal{H}'_{l,s})_K$ *is unitarizable if and only if one of the following two cases occurs:*

   (a) $s=i\nu$, $\nu\in\mathbf{R}$.

   (b) $s\in(2l-\rho+2,-2l+\rho-2)$.

Case (b) corresponds to the complementary series for $\mathrm{Sp}(1,n)$. *They exist if and only if* $2l<2n-1$.

The fact that τ_l occurs exactly once in $T'_{l,s}|_K$ for the irreducible $T'_{l,s}$ is known a priori ([Go], Corollary to Theorem 8, p. 522; [Dei], Theorem 3). The explicit K-module decomposition of $(\mathcal{H}'_{l,s})_K$ in [HT], pp. 53–54, shows that this is actually true for all the $T'_{l,s}$. The K-submodule of $(\mathcal{H}'_{l,s})_K$ equivalent to τ_l is the only element in the "fiber of K-types" over the point $(0,2l)$ in Diagrams 5.10 and 5.14 of [HT]. It is contained in a unique subquotient of $T'_{l,s}$, which can then be located in the diagrams used to determine the unitarizability of the various subquotients ([HT], pp. 25 and 30). We therefore obtain the following proposition.

5.4. PROPOSITION. *Suppose* $(\mathcal{H}'_{l,s})_K$ *is a reducible* $\mathfrak{U}(\mathfrak{g})$*-module and assume* $s\geq 0$. *The irreducible subquotient of* $(\mathcal{H}'_{l,s})_K$ *in which* τ_l *occurs is unitarizable if and only if* $s\equiv 2(l-n)+1 \pmod 2$ *and* $2l>s-\rho+4n-2$. *That is, if and only if* $2l\geq 2n-1$ *and* $s\in\{s_j=2(l-n-j)+1 : j=0,1,\dots; s_j\geq 0\}$.

Let $(T_{l,s},\mathcal{H}_{l,s})$ denote the subquotient representation of $T'_{l,s}$ corresponding to the irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ in which τ_l occurs. Then $T_{l,s}$ is an admissible Hilbert representation of $\mathrm{Sp}(1,n)$, and $T_{l,s}(g)v=T'_{l,s}(g)v$ for all $v\in\mathcal{H}'_{l,s}(\tau_l)$. Lemma 5.2 yields

5.5. COROLLARY. *Let $E(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto the K-isotypic subspace of type τ_l. Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)T_{l,s}(g)]\,. \tag{5.43}$$

$(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM. *$\zeta_{l,s} = \zeta_{l,-s}$ is positive definite if and only if one of the following cases occurs:*

1. $s = i\nu,\ \ \nu \in \mathbf{R}$.
2. *If* $2l \geq 2n-1$: $\pm s = s_j := 2(l-n-j)+1$ *for integers* $j \geq 0$ *so that* $s_j > 0$. (*discrete series*)
3. *If* $2l < 2n-1$: $s \in (2l-\rho+2, -2l+\rho-2)$. (*complementary series*)

The situation for s real and nonnegative is represented in Figure 6.1.

6. THE τ_l-ABEL TRANSFORM

Proposition 3.2 proves that the τ_l-Abel transform is a *-homomorphism of $\mathcal{D}(G;\chi_l)$ into the convolution algebra $\mathcal{D}_+(\mathbf{R})$ consisting of the even C^∞ functions on $\mathbf{R}$ with compact support. The main theorem of this section states that the τ_l-Abel transform is also a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$, and gives a formula for its inverse.

Identify A with $\mathbf{R}$ under the map $t \mapsto a_t$. Restriction to A then identifies $\mathcal{D}(G;\chi_l)$ with $\mathcal{D}_+(\mathbf{R})$. Let $\mathcal{D}([1,\infty))$ denote the set of the compactly supported C^∞ functions on $[1,\infty)$ (right differentiability at 1 is considered). Define a map H by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$

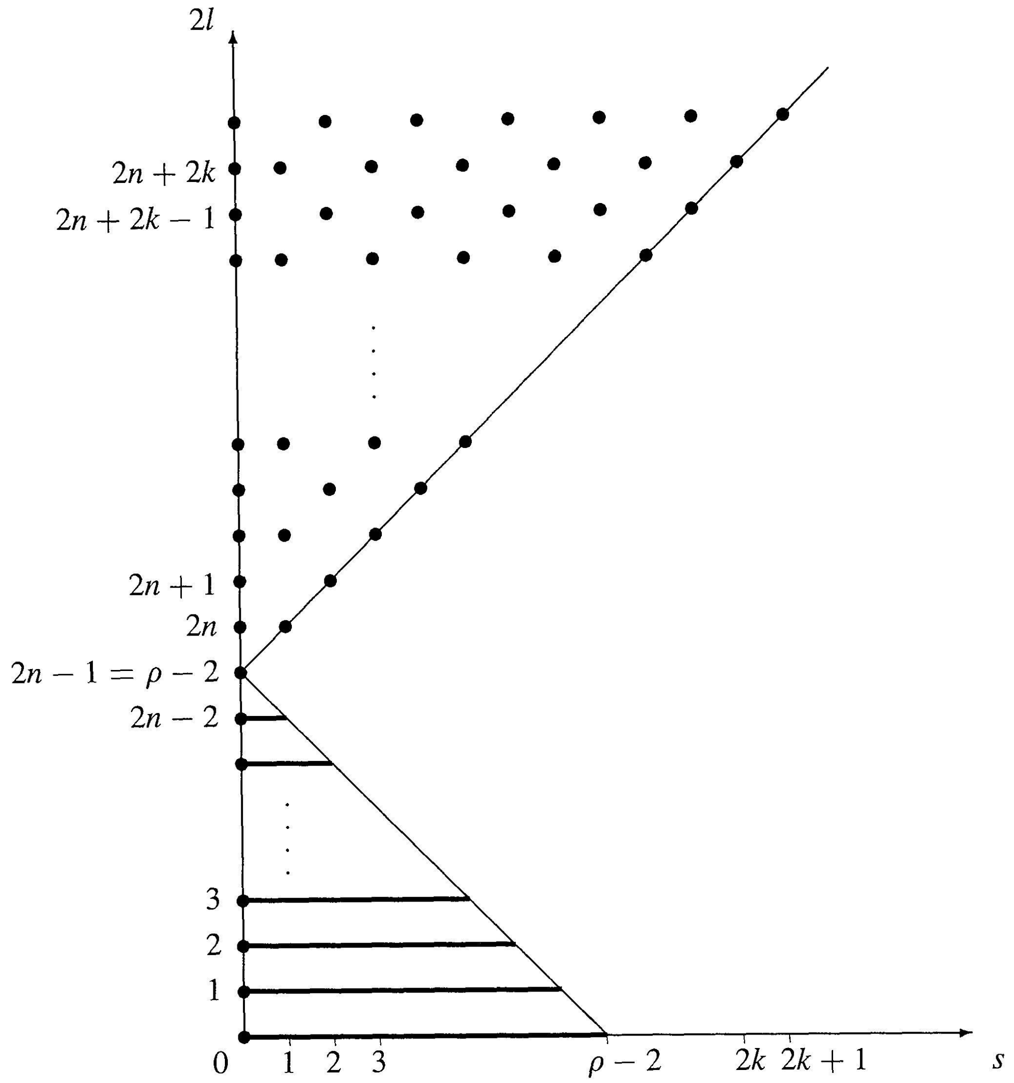

FIGURE 6.1
Positive definite $\zeta_{l,s}$ for real $s\geq 0$

for $f\in\mathcal{D}(G;\chi_l)$. Lemma 2 and its corollary in [Rou] imply

6.1. LEMMA. *H is a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}([1,\infty))$.*

For every $\mu\in\mathbf{C}$ with $\Re\mu>0$, the Weyl fractional integral transform of $\varphi\in\mathcal{D}([1,\infty))$ is defined by

$$\mathcal{W}_\mu\varphi(x)=\frac{1}{\Gamma(\mu)}\int_x^\infty \varphi(u)(u-x)^{\mu-1}\,du\,,\qquad x\in[1,\infty)\,. \tag{6.44}$$

Analytic continuation of $\mathcal{W}_\mu$ to $\Re\mu\leq 0$ is obtained via repeated integration by parts of (6.44): for every integer $m\geq 0$

$$\mathcal{W}_\mu\varphi(u)=\frac{(-1)^m}{\Gamma(\mu+m)}\int_u^\infty \frac{d^m\varphi}{dx^m}(x)\,(x-u)^{\mu+m-1}\,dx\,.$$

For every integer $m \geq 0$, the Gegenbauer transform (of dimension 4) of $\varphi \in \mathcal{D}([1,\infty))$ is defined by

$$\text{(6.45)} \qquad \mathcal{G}_m\varphi(u) = \frac{4\pi}{m+1} \int_u^\infty \varphi(x)\, C_m^1\left(\frac{u}{x}\right)(x^2-u^2)^{\frac{1}{2}} x\, dx\,, \quad u \in [1,\infty)\,,$$

where

$$\text{(6.46)} \qquad C_m^1(y) = (m+1)F\Big(-m, m+2; \frac{3}{2}; \frac{1-y}{2}\Big)$$

is the Gegenbauer polynomial of indices $(1,m)$ (cf. e.g. [E$^+$], 3.15 (3)).

6.2. LEMMA ([K1], Theorem 3.2; [Dea], Formulas (28) and (29)).

1. *For every $\mu \in \mathbf{C}$, $\mathcal{W}_\mu$ is a bijection of $\mathcal{D}([1,\infty))$ onto itself. The inverse mapping of $\mathcal{W}_\mu$ is $\mathcal{W}_{-\mu}$.*

2. *For every integer $m \geq 0$, $\mathcal{G}_m$ is a bijection of $\mathcal{D}([1,\infty))$ onto itself. The inverse mapping of $\mathcal{G}_m$ is given by*

$$\text{(6.47)} \qquad \mathcal{G}_m^{-1}\psi(x) = -\frac{1}{2\pi^2(m+1)}\frac{1}{x^2} \int_x^\infty \frac{d^3\psi}{du^3}(u)\, C_m^1\left(\frac{u}{x}\right)(u^2-x^2)^{\frac{1}{2}}\, du$$

for all $\psi \in \mathcal{D}([1,\infty))$ and all $x \in [1,\infty)$.

6.3. THEOREM. *The τ_l-Abel transform is a bijection of $\mathcal{D}(G;\chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$. It can be written as the composition*

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2}\, H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H\,,$$

and its inverse is given by

$$\mathcal{A}_l^{-1} = \frac{d_l^2}{(2\pi)^{2(n-1)}}\, H^{-1} \circ \mathcal{G}_{2l}^{-1} \circ \mathcal{W}_{2-2n} \circ H\,.$$

Moreover, the support of the restriction to $A \equiv \mathbf{R}$ of $f \in \mathcal{D}(G;\chi_l)$ is contained in $[-R,R]$ if and only if the support of $\mathcal{A}_l f$ is contained in $[-R,R]$.

Proof. Identify the set of pure quaternions $w = i\,b + j\,c + k\,d \in \mathbf{H}$ with $\mathbf{R}^3$, and $\mathbf{H}^{n-1}$ with $\mathbf{R}^{4(n-1)}$. If $z \in \mathbf{H}^{n-1}$, then $-[z,z] \equiv |z|^2$ is the square of the Euclidean norm of z in $\mathbf{R}^{4(n-1)}$. For $a_t \in A$ and $n = n(w,z) \in N$ we have

$$(a_t n)_{00} = \cosh t + e^t(w + \tfrac{1}{2}|z|^2)\,.$$

Let $f \in \mathcal{D}(G;\chi_l)$. Applying Lemma 3.3 and Formulas (1.5), we obtain

$$\mathcal{A}_l f(t) = \frac{1}{d_l^2} e^{\rho t} \int_N f(a_t n)\, dn$$

$$= \frac{1}{d_l^3} e^{\rho t} \int_N \chi_l\left(\frac{(a_t n)_{00}}{|(a_t n)_{00}|}\right) Hf(|(a_t n)_{00}|)\, dn$$

$$= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} \chi_l\left(\frac{\cosh t + e^t(w + \frac{1}{2}|z|^2)}{|\cosh t + e^t(w + \frac{1}{2}|z|^2)|}\right)$$

$$\times\, Hf(|\cosh t + e^t(w + \tfrac{1}{2}|z|^2)|)\, dz\, dw$$

$$= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1\left(\frac{\cosh t + \frac{1}{2}e^t|z|^2}{[(\cosh t + \frac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}}}\right)$$

$$\times\, Hf([(\cosh t + \tfrac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}})\, dz\, dw$$

(by Formula (4.40))

$$= \frac{4^{n-1}}{d_l^3} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1\left(\frac{\cosh t + |X|^2}{[(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}}}\right)$$

$$\times\, Hf([(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}})\, dX\, dY$$

(by substituting $X = \frac{1}{\sqrt{2}} e^{\frac{t}{2}} z,\ Y = e^t w$)

$$= \frac{2^\rho}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_0^\infty \int_0^\infty C_{2l}^1\left(\frac{\cosh t + r^2}{[(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}}}\right)$$

$$\times\, Hf([(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}})\, r^{4n-5} s^2\, ds\, dr$$

(by passing to spherical coordinates in $\mathbf{R}^{4(n-1)}$ and in $\mathbf{R}^3$)

$$= \frac{2^{2n}}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[\int_0^{\infty} C_{2l}^1\left(\frac{u}{[u^2+s^2]^{\frac{1}{2}}}\right) Hf([u^2+s^2]^{\frac{1}{2}})\, s^2\, ds \right] \times (u-\cosh t)^{2n-3}\, du$$

(by setting $u = \cosh t + r^2$)

(6.48)

$$= \frac{2^{2n}\pi^{2n-1}}{d_l^3\Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[\int_u^{\infty} C_{2l}^1\left(\frac{u}{x}\right) Hf(x)\,(x^2-u^2)^{\frac{1}{2}} x\, dx \right] (u-\cosh t)^{2n-3} du$$

(by setting $x = [u^2+s^2]^{\frac{1}{2}}$)

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2\Gamma(2n-1)} \int_{\cosh t}^{\infty} (\mathcal{G}_{2l}Hf)(u)\,(u-\cosh t)^{2n-3}\, du$$

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2} \mathcal{W}_{2n-2}\, \mathcal{G}_{2l}\, Hf(\cosh t)$$

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1}\, \mathcal{W}_{2n-2}\, \mathcal{G}_{2l}\, Hf)(t)\,,$$

i.e.

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2}(H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H)\,.$$

The inversion formula immediately follows from Lemma 6.2.

The restriction to $A \equiv \mathbf{R}$ of $f \in \mathcal{D}(G;\chi_l)$ has its support $\operatorname{supp} f$ contained in $[-R,R]$ if and only if $\operatorname{supp} Hf \subset [1,\cosh R]$. Moreover, if $\operatorname{supp}\varphi \subset [1,\cosh R]$, then $\operatorname{supp}\mathcal{W}_\mu\varphi$, $\operatorname{supp}\mathcal{G}_m\varphi$ and $\operatorname{supp}\mathcal{G}_m^{-1}\varphi$ are also contained in $[1,\cosh R]$. The last statement then follows from the formulas for $\mathcal{A}_l$ and $\mathcal{A}_l^{-1}$. $\square$

The τ_l*-spherical transform* of $f \in \mathcal{D}(G;\chi_l)$ is the function $\widehat{f}_l$ on $\mathbf{C}$ defined by

$$\widehat{f}_l(s) = \int_G f(g)\zeta_{l,s}(g)\,dg, \qquad s \in \mathbf{C}.$$

Let $\mathcal{S}_l\colon f \mapsto \widehat{f}_l$ denote the τ_l-spherical transform, and let $\mathcal{F}$ denote the Fourier-Laplace transform on $\mathbf{R}$. Formulas (3.20) and (3.22) yield

$$\mathcal{S}_l = \mathcal{F} \circ \mathcal{A}_l. \tag{6.49}$$

Let $\mathcal{H}_+^R(\mathbf{R})$ denote the set of even functions h on $\mathbf{C}$ which are entire rapidly decreasing functions of exponential type R: for every integer $N \geq 0$ there is a constant $C_N > 0$ so that

$$|h(s)| \leq C_N(1+|s|)^{-N}e^{R|\Re s|} \quad \text{for all } s \in \mathbf{C}.$$

Set $\mathcal{H}_+(\mathbf{R}) := \bigcup_{R>0}\mathcal{H}_+^R(\mathbf{R})$. Theorem 6.3 and the Paley-Wiener Theorem for the Fourier-Laplace transform of even functions on $\mathbf{R}$ prove the following theorem.

6.4. THEOREM (Paley-Wiener Theorem). *The* τ_l*-spherical transform is a bijection of* $\mathcal{D}(G;\chi_l)$ *onto* $\mathcal{H}_+(\mathbf{R})$. *Moreover, the restriction of* $f \in \mathcal{D}(G;\chi_l)$ *to* $A \equiv \mathbf{R}$ *has support in* $[-R,R]$ *if and only if* $\widehat{f}_l \in \mathcal{H}_+^R(\mathbf{R})$.

We conclude this section by observing that the τ_l-Abel transform is related, as one should expect, to the Abel transform of [K2], §5.

Reversing the order of integration and substituting $x = \cosh\tau$ and $u = \cosh w$, we obtain from (6.48)

$$\mathcal{A}_l f(t) = \int_t^\infty A_l(t,\tau) f(\tau)\,d\tau \tag{6.50}$$

where

$$A_l(t,\tau) := \frac{(2\pi)^{2n-1}}{d_l^3\Gamma(2n-2)}\sinh(2\tau)\int_t^s C_{2l}^1\left(\frac{\cosh w}{\cosh\tau}\right)(\cosh^2\tau - \cosh^2 w)^{\frac{1}{2}}$$
$$\times (\cosh w - \cosh t)^{2n-3}\sinh w\,dw.$$

Substituting also $y = \dfrac{\cosh\tau - \cosh w}{\cosh\tau - \cosh t}$ and setting

$$\gamma(t,\tau) = \frac{\cosh\tau - \cosh t}{2\cosh\tau} \qquad \text{and} \qquad K_l = \frac{(2\pi)^{2n-1}}{d_l^3\Gamma(2n-2)},$$

we get from Formula (6.46)

$$A_l(t,\tau) = \sqrt{2}\, K_l \sinh(2\tau)(\cosh\tau - \cosh t)^{2n-\frac{3}{2}}(\cosh\tau)^{\frac{1}{2}}$$
$$\times \int_0^1 C^{1}_{2l}(1-2\gamma(t,\tau)y))y^{\frac{1}{2}}(1-y)^{2n-3}(1-\gamma(t,\tau)y)^{\frac{1}{2}}\,dy$$
$$= \sqrt{2}(2l+1)K_l \sinh(2\tau)(\cosh\tau - \cosh t)^{2n-\frac{3}{2}}(\cosh\tau)^{\frac{1}{2}}$$
$$\times \int_0^1 F\Big(\frac{3}{2}+2l, -2l-\frac{1}{2}; \frac{3}{2}; \gamma(t,\tau)y\Big) y^{\frac{1}{2}}(1-y)^{2n-3}(1-\gamma(t,\tau)y)^{\frac{1}{2}}dy\,.$$

If we now apply the relation ([E$^+$], 2.9(2))

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z) \tag{6.51}$$

and Bateman's Formula ([E$^+$], 2.4(2))

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)} \int_0^1 x^{s-1}(1-x)^{c-s-1}F(a,b;s;xz)\,dx \tag{6.52}$$

for $\Re c > \Re s > 0, z \neq 1, |\arg(1-z)| < \pi$,

we finally obtain

$$A_l(t,\tau) = \frac{(2\pi)^{2n-\frac{1}{2}}}{2\Gamma(2n-\frac{1}{2})}\frac{1}{d_l^2} \sinh(2\tau)(\cosh\tau - \cosh t)^{2n-\frac{3}{2}}(\cosh\tau)^{\frac{1}{2}} \tag{6.53}$$
$$\times F\left(\frac{3}{2}+2l, -2l-\frac{1}{2}, 2n-\frac{1}{2}, \gamma(t,\tau)\right).$$

The comparison of Formula (6.53) with the kernel $A_{2n-1,2l+1}(t,\tau)$ in [K2], Formula (5.60), gives

(6.54)

$$A_l(t,\tau) = \frac{1}{2}\frac{1}{{d_l}^2}\left(\frac{\pi}{4}\right)^{2n}\frac{1}{\Gamma(2n)} 2^{-4l}(\cosh\tau)^{-2l} A_{2n-1,2l+1}(t,\tau)$$
$$= 2^{-4l}C_l(\cosh\tau)^{-2l} A_{2n-1,2l+1}(t,\tau)$$

where we have set

$$C_l := \frac{1}{2}\frac{1}{(2l+1)^2}\left(\frac{\pi}{4}\right)^{2n}\frac{1}{\Gamma(2n)}\,. \tag{6.55}$$

7. THE PLANCHEREL FORMULA

Let L_l denote the differential operator

$$L_l = \frac{d^2}{dt^2} + [(4n-1)\coth t + 3\tanh t]\frac{d}{dt} + \frac{4l(l+1)}{\cosh^2 t}\,. \tag{7.56}$$

Proposition 4.3, Part 3, and Formula (4.39) prove that the restriction to $A \equiv \mathbf{R}$ of the τ_l-spherical function $\zeta_{l,s}$ is the unique solution to the differential equation

$$L_l u = (s^2 - \rho^2)u \tag{7.57}$$

satisfying $u(0) = 1$ and $u'(0) = 0$. If s is not an integer, two linearly independent solutions of (7.57) are also the functions $\Phi_{l,\pm s}$ defined by

$$\Phi_{l,\pm s}(t) = (2\cosh t)^{\pm s-\rho} F\left(\frac{\rho \mp s}{2} + l, \frac{\rho \mp s}{2} - l - 1; 1 \mp s; 1 - \tanh^2 t\right). \tag{7.58}$$

$\zeta_{l,s}$ is an even function of $s \in \mathbf{C}$. Therefore if $s \in \mathbf{C} \setminus \mathbf{Z}$ there exists a constant $c_l(s)$ so that

$$\zeta_{l,s} = c_l(s)\Phi_{l,s} + c_l(-s)\Phi_{l,-s}\,. \tag{7.59}$$

Formula 2.9(33) in [E$^+$] gives $c_l(s)$ explicitly as the meromorphic function

$$c_l(s) = 2^{\rho-s} \frac{\Gamma(2n)\Gamma(s)}{\Gamma\left(\frac{\rho+s}{2} + l\right)\Gamma\left(\frac{\rho+s}{2} - l - 1\right)}\,. \tag{7.60}$$

The function c_l determines the Plancherel measure for the τ_l- spherical transform. We immediately give the information that will make this measure explicit.

On the domain $S = \{s \in \mathbf{C} : \Re s \geq 0\}$, the function $c_l(s)$ has a simple pole at $s = 0$ and, if $2l \geq 2n - 1$, $c_l(s)$ also has simple zeros at the points of the set

$$D_l = \{s_j = 2(l-j+1) - \rho = 2(l-j-n)+1 : j = 0, 1, \dots \text{ and } s_j > 0\}\,. \tag{7.61}$$

The singularity of c_l^{-1} at $-s_j$ can be removed by setting

$$\begin{aligned} c_l(-s_j) :=& 2^{\rho+s_j} \frac{\Gamma(2n)}{\Gamma\left(\frac{\rho-s_j}{2} + l\right)} \frac{\operatorname*{Res}_{s=s_j}\Gamma(-s)}{\operatorname*{Res}_{s=s_j}\Gamma\left(\frac{\rho-s}{2} - l - 1\right)} \\ =& 2^{\rho+s_j}(-1)^j \frac{\Gamma(2n)\Gamma(2(l-n+1)-j)}{\Gamma(2n+j)\Gamma(2(l-n-j+1))}\,. \end{aligned} \tag{7.62}$$

Since

$$\operatorname*{Res}_{s=s_j}\left[\frac{1}{c_l(s)}\right] = 2^{-\rho-s_j}\,(-1)^j\,\frac{\Gamma(2l+1-j)}{\Gamma(2n)\Gamma(2(l-n-j)+1)\Gamma(j+1)}\,,$$

we obtain from (7.62)

(7.63)

$$\operatorname*{Res}_{s=s_j}\left[\frac{1}{c_l(s)c_l(-s)}\right] = \frac{2^{-2\rho}}{\Gamma(2n)^2}\,\frac{\Gamma(j+2n)}{\Gamma(j+1)}\,\frac{\Gamma(2l+1-j)}{\Gamma(2(l-n-j)+1)}[2(l-n-j)+1]\,.$$

Setting $\alpha = 2n-1$, we can rewrite these residues in terms of the shifted factorials

$$(a)_\alpha := a(a+1)\cdots(a+\alpha-1)$$

as

$$\text{(7.64)}\quad \operatorname*{Res}_{s=s_j}\left[\frac{1}{c_l(s)c_l(-s)}\right] = 2^{-2\rho}\,[2(l-n-j)+1]\,\frac{(j+1)_\alpha(2(l-n+1)-j)_\alpha}{(\alpha!)^2}\,.$$

Moreover, if $s \in \mathbf{R}$, then $c_l(-is) = \overline{c_l(is)}$ and

$$\text{(7.65)}\quad |c_l(is)|^{-2} = \frac{2^{-2\rho}}{\pi\Gamma(2n)^2}\,s\sinh(\pi s)\left|\Gamma\Big(\frac{\rho+is}{2}+l\Big)\right|^2\left|\Gamma\Big(\frac{\rho+is}{2}-l-1\Big)\right|^2.$$

We have let the parameter l range in the set $\mathbf{N}/2$. Nevertheless, the explicit formulas for the functions $\zeta_{l,s}$, $\Phi_{l,s}$ and c_l allow us to consider l as a complex variable. Indeed, Formulas (4.39), (7.58) and (7.60) show, respectively, that for every fixed $t \in \mathbf{R}$, $\zeta_{l,s}$ is holomorphic in $(l,s) \in \mathbf{C}^2$, for fixed $t > 0$, $\Phi_{l,s}$ is holomorphic in $(l,s) \in \mathbf{C}^2$, and that $c_l(s)$ is an entire function of $l \in \mathbf{C}$ and a meromorphic function of $s \in \mathbf{C}$.

Because of Remark 2.3 and Formula (1.6), for every $f \in \mathcal{D}(G;\chi_l)$ we have

$$\text{(7.66)}\qquad \widehat{f}_l(s) = C_l\int_0^\infty f(a_t)\zeta_{l,s}(t)\Delta(t)\,dt$$

where C_l is given by (6.55) and $f(t) := f(a_t)$. Formula (7.66) can be employed to define $\widehat{f}_l$ for every $l \in \mathbf{C}$ and every $f \in \mathcal{D}_+(\mathbf{R})$. Morera's Theorem then implies that $\widehat{f}_l(s)$ is a holomorphic function of $(l,s) \in \mathbf{C}^2$.

Lemmas 2.1, 2.2 and 2.3 in [K1] prove:

1. For each $r > 0$ there is a constant $K_r > 0$ such that

$$\text{(7.67)}\qquad |c_l(s)|^{-1} \leq K_r(1+|s|)^{2n-\frac{1}{2}}$$

if $\Re s \geq 0$ and $c_l(s') \neq 0$ for $|s'-s| \leq r$.

2. For each $\delta > 0$ there is a constant $K_\delta > 0$ such that

$$|\Phi_{l,-s}(t)| \le K_\delta e^{-(\Re s+\rho)t} \tag{7.68}$$

if $\Re s \ge 0$ and $t \ge \delta > 0$.

3. There exists a constant $K > 0$ such that

$$|\zeta_{l,s}(t)| \le K(1+t)e^{(|\Re s|-\rho)t}$$

for all $t \ge 0$ and all $s \in \mathbf{C}$.

L_l is a symmetric operator on the space $L^2(\Delta(t)\,dt)$ of functions on $(0,\infty)$ which are L^2-integrable with respect to the measure $\Delta(t)\,dt$. Greeen's Formula and Equation (7.57) satisfied by $\zeta_{l,s}$ give for every $l, s \in \mathbf{C}$ and $f \in \mathcal{D}_+(\mathbf{R})$

$$(L_l^n f)_l^\wedge(s) = (s^2-\rho^2)^n \widehat{f}_l(s)$$

where $L_l^n = \underbrace{L_l \circ \cdots \circ L_l}_{n \text{ times}}$. Note that, for $k \ge 2n$, $L_l^k f(t)\,\Delta(t)$ is continuous and compactly supported on $[0,\infty)$. If $\operatorname{supp} f \subset [-R,R]$, then (7.68) gives

$$|(s^2-\rho^2)^n \widehat{f}_l(s)| \le \int_0^\infty |L_l^n f(t)|\zeta_{l,s}(t)\Delta(t)\,dt \le Ce^{|\Re s|R} \tag{7.69}$$

for some constant $C > 0$. The above estimates allow us to conclude: for every $r > 0$ and $\delta > 0$ there exists a constant $K_{r,\delta} > 0$ so that

$$\left|\widehat{f}_l(\mu+i\nu)\frac{\Phi_{l,-\mu+i\nu}(t)}{c_l(\mu+i\nu)}\right| \le K_{r,\delta}(1+|\mu+i\nu|)^{2n-\frac{1}{2}-2n}e^{(R-t)\mu-\rho t} \tag{7.70}$$

if $\mu \ge 0$, $c_l(s') \ne 0$ for $|s'-(\mu+i\nu)| \le r$, and $t \ge \delta > 0$.

Let D_l be as in (7.61), and set

$$\mu_0 := \begin{cases} \max D_l = 2(l-n)+1 & \text{if } 2l \ge 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\mu_1 > \mu_0$ and define for $t > 0$

$$(\widehat{f}_l)_l^\vee(t) := \frac{1}{2\pi C_l}\int_{-\infty}^{\infty} \widehat{f}_l(\mu_1+i\nu)\Phi_{l,-\mu_1-i\nu}(t)\,\frac{d\nu}{c_l(\mu_1+i\nu)}$$

where C_l is given by (6.55). Observe that the integrand is a meromorphic function of $\mu+i\nu \in \mathbf{C}$ with singularities given by those of c_l^{-1}. Because of (7.70), Cauchy's Theorem gives

$$(\widehat{f}_l)_l^\vee(t) = \frac{1}{2\pi C_l}\int_{-\infty}^{+\infty} \widehat{f}_l(\mu+i\nu)\Phi_{l,-\mu-i\nu}(t)\,\frac{d\nu}{c_l(\mu+i\nu)} \tag{7.71}$$

for every $\mu > \mu_0$. Letting $\mu \to +\infty$, we find that $(\widehat{f}_l)_l^\vee(t) = 0$ for $t > R$.

7.1. THEOREM (Inversion formula, first form). *For every* $l \in \mathbf{C}$, $f \in \mathcal{D}_+(\mathbf{R})$ *and* $t > 0$

$$f(t) = (\widehat{f}_l)_l^\vee(t)\,. \tag{7.72}$$

Proof. Since $(\widehat{f}_l)_l^\vee$ is an entire function of l, it is enough to establish (7.72) when $0 \leq 2l < 2n-1$. In this case, (7.71) holds also for $\mu = 0$. We then have to prove that for all $t > 0$

$$f(t) = \frac{1}{2\pi C_l} \int_{-\infty}^{\infty} \widehat{f}_l(i\nu)\zeta_{l,i\nu}(t)\, \frac{d\nu}{|c_l(i\nu)|^2}\,. \tag{7.73}$$

The method of Gangolli-Helgason-Rosenberg applies to this purpose without essential modifications. We therefore only sketch the proof, and refer to [Ros] and to [GV], § 6.6, for the details.

Endow $\mathcal{D}(\mathbf{R})$ with the usual inductive limit topology, and consider $\mathcal{D}_+(\mathbf{R})$ with the induced topology. Then the assignment

$$T(f) = \frac{1}{2\pi C_l} \int_{-\infty}^{\infty} \widehat{f}_l(i\nu)\, \frac{d\nu}{|c(i\nu)|_l^2}\,, \qquad f \in \mathcal{D}_+(\mathbf{R}),$$

defines a distribution T on $\mathcal{D}_+(\mathbf{R})$. As in the case of K-bi-invariant functions, it is possible to show that T is indeed a measure, and hence there is a constant C so that

$$Tf = Cf(e) \qquad \text{for all } f \in \mathcal{D}_+(\mathbf{R})\,. \tag{7.74}$$

If $f \in \mathcal{D}_+(\mathbf{R})$, so does its generalized translate

$$T_\tau f(t) = \int_0^\infty f(u) K_l(t,\tau,u)\Delta(u)\, du\,,$$

where the kernel $K_l(t,\tau,u)$ is as in (4.42). $T_\tau f$ satisfies $T_\tau f(t) = T_t f(\tau)$, $T_0 f = f$ and, because of (4.41),

$$(T_\tau f)_l^\wedge(s) = \zeta_{l,s}(\tau)\widehat{f}_l(s), \qquad s, l \in \mathbf{C},\ \tau \in \mathbf{R}$$

Since $T_t f(0) = f(t)$, Formula (7.72) follows from (7.73) provided $C = 1$, which can be proven true in the same lines of [Ros], p. 147. $\square$

7.2. THEOREM (Inversion formula, second form). *Let D_l be the set defined by Formula (7.61) if $2l \geq 2n-1$ and $D_l = \varnothing$ otherwise. Let C_l be the constant in (6.55). For every $f \in \mathcal{D}(G;\chi_l)$ and $g \in G$,*

$$(7.75)\quad f(g) = \frac{1}{C_l}\sum_{s_j\in D_l}\left[\operatorname*{Res}_{s=s_j}\frac{1}{c_l(s)c_l(-s)}\right]\widehat{f}_l(s_j)\zeta_{l,s_j}(g) + \frac{1}{2\pi C_l}\int_0^\infty \widehat{f}_l(is)\zeta_{l,is}(g)\,\frac{ds}{|c_l(is)|^2}\,.$$

Proof. Fix $\mu > \mu_0$, and let γ_R be the rectangular contour of vertices $\pm iR$ and $\mu \pm iR$. Integrating the function

$$(7.76)\qquad s \longmapsto \frac{1}{2\pi C_l}\frac{\widehat{f}_l(s)\Phi_{l,-s}(t)}{c_l(s)}$$

along γ_R and letting $R\to\infty$, we obtain from (7.70), (7.61) and Theorem 7.1

$$f(a_t) = \frac{1}{C_l}\sum_{s_j\in D_l}\left[\operatorname*{Res}_{s=s_j}\frac{\Phi_{l,-s}(t)}{c_l(s)}\right]\widehat{f}_l(s_j) + \frac{1}{2\pi C_l}\int_{-\infty}^{+\infty}\widehat{f}_l(is)\Phi_{l,-s}(t)\frac{ds}{c_l(is)}$$

for all $t > 0$. Equation (7.59) therefore proves the claim for $g = a_t$, $t > 0$, and hence for all t by continuity and evenness of the functions on both sides of (7.75). Formula (7.75) thus holds for arbitrary $g \in G$ because of (2.14) and (3.24). □

Since $\overline{\zeta_{l,s}} = \zeta_{l,s}$, the usual trick of replacing f with $f^* * f$ in (7.75) evaluated at e gives the Plancherel formula.

7.3. THEOREM (Plancherel Theorem). *Let D_l and C_l be as in Theorem 7.1. Define a measure σ_l on $i\mathbf{R}_+ \cup D_l$ by*

$$(7.77)\quad \int_{i\mathbf{R}_+\cup D_l} g(s)\,d\sigma_l(s) = \frac{1}{C_l}\sum_{s_j\in D_l}\left[\operatorname*{Res}_{s=s_j}\frac{1}{c_l(s)c_l(-s)}\right] g(s_j) + \frac{1}{2\pi C_l}\int_0^\infty g(is)\,\frac{ds}{|c_l(is)|^2}\,.$$

Let $L^2(G;\chi_l)$ denote the closure of $\mathcal{D}(G;\chi_l)$ in $L^2(G)$, and let $L^2(d\sigma_l)$ be the space of L^2-integrable functions on $i\mathbf{R}_+ \cup D_l$ with respect to the measure $d\sigma_l$. Then the map $f \mapsto \widehat{f}_l$ extends to an isometric isomorphism of $L^2(G;\chi_l)$ onto $L^2(d\sigma_l)$:

$$(7.78)\qquad \int_G |f(g)|^2\,dg = \int_{i\mathbf{R}_+\cup D_l}|\widehat{f}_l(s)|^2\,d\sigma_l(s)\,.$$

The techniques employed to prove the inversion formula (that is, Koornwinder's analytic continuation argument and the change of contour of integration) are the same used in [Shi] for the case of Hermitian symmetric pairs. Our choice of this method of proof is motivated by the propaedeutic nature of this paper. In fact, the computations involved in the proofs presented above are very much in the spirit of those required for the decomposition of the canonical representations in [DP].

We just mention a few alternative methods. First of all, because of Formula (7.74) and Part 3 of Proposition 4.3, the spectral theorem for the τ_l-spherical transform can be deduced from the spectral theorem for the differential operator L_l (see (7.56)) on a suitable domain in $L^2(\Delta(t)\,dt)$ on which it is self-adjoint. The latter theorem can be classically determined as an application of the Weyl-Titchmarsh Theorem. A second method is obtained observing the relation, ensured by Formula (4.39), between the τ_l-spherical transform and the Jacobi transform. Theorems 2.3 and 2.4 of [K2] are then quickly translated to our situation. Finally, observe that Koornwinder's method with the Abel transform can also be applied directly here because of Formula (6.54).

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