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3. THE τ_l-SPHERICAL FUNCTIONS

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}=\mathfrak{sp}(1,n)$. The elements of $\mathfrak{U}(\mathfrak{g})$ are considered as left-invariant differential operators on $G=\mathrm{Sp}(1,n)$ acting on C^∞ functions on the right:

$$f(g;X):=\frac{d}{dt}f(g\exp tX)\Big|_{t=0}\qquad (f\in C^\infty(G),\ X\in\mathfrak{g},\ g\in G).$$

We adopt Harish-Chandra's notation $f(g;D)$ for the image of $f\in C^\infty(G)$ under the right action of $D\in\mathfrak{U}(\mathfrak{g})$. The set of K-invariant elements of $\mathfrak{U}(\mathfrak{g})$ is denoted by $\mathfrak{U}(\mathfrak{g})^K$.

3.1. THEOREM ([Go], Theorems 8, 10 and 14; [GaV], Theorems 1.3.14, 1.4.5 and Proposition 1.4.4). *Let $l\in\mathbf{N}/2$ be fixed. Let ζ be a complex-valued continuous function on G satisfying (2.9), (2.10) and $\zeta(e)=1$ (e is the unit element of G). The following statements are mutually equivalent.*

1. *The mapping $f\mapsto\int_G f(g)\zeta(g)\,dg$ is an algebra homomorphism of $\mathcal{D}(G;\chi_l)$ into $\mathbf{C}$.*
2. *ζ satisfies the functional equation*

$$\int_K\zeta(kg_1k^{-1}g_2)\,dk=\zeta(g_1)\zeta(g_2)\tag{3.15}$$

   for all $g_1,g_2\in G$.
3. *ζ is a common eigenfunction of the elements of $\mathfrak{U}(\mathfrak{g})^K$.*

A function ζ satisfying the equivalent conditions of Theorem 3.1 is called a *spherical function of type τ_l (and height 1)* or briefly a *τ_l-spherical function* [1]).

Observe that Condition 3 implies in particular that every τ_l-spherical function is analytic on G because $\mathfrak{U}(\mathfrak{g})^K$ contains an elliptic differential operator.

[1]) Our definition of τ_l-spherical function ζ is formally less restrictive than Godement's, which also requires ζ to be quasi-bounded with respect to some seminorm on G (cf. [Go], p. 519). It is possible to show (e.g. [GaV], Theorem 1.3.14) that each of the conditions for ζ given by Theorem 3.1 is equivalent to the existence of an irreducible Fréchet representation $(T,\mathcal{H})$ – with a d_l-dimensional K-isotypic subspace $\mathcal{H}(\tau_l)$ of type τ_l – for which

$$\zeta=\frac{1}{d_l}\,\mathrm{tr}[E(\tau_l)TE(\tau_l)]\,,\tag{$*$}$$

$E(\tau_l)$ being the projection of $\mathcal{H}$ onto $\mathcal{H}(\tau_l)$. $(T,\mathcal{H})$ can be chosen to be a Banach representation of G if and only if ζ is quasi-bounded with respect to some seminorm on G. In Section 5 we will determine, for each τ_l-spherical function ζ, an irreducible *Hilbert* representation $(T,\mathcal{H})$ for which $(*)$ holds. It follows, in particular, that the condition of quasi-boundedness is, in our case, automatically satisfied.

For complex-valued functions f on G and F on $\mathbf{R}$, we set

$$f^*(g) = \overline{f(g^{-1})} \qquad (g \in G)$$
$$F^*(t) = \overline{F(-t)} \qquad (t \in \mathbf{R}).$$

The *τ_l-Abel transform* of $f \in \mathcal{D}(G;\chi_l)$ is the function $\mathcal{A}_l f$ on $\mathbf{R}$ defined by

$$\mathcal{A}_l f(t) = \frac{1}{d_l^2}\, e^{\rho t} \int_N f(a_t n)\, dn. \tag{3.16}$$

Its properties are summarized in the following proposition.

3.2. PROPOSITION. *For all $f \in \mathcal{D}(G;\chi_l)$, $\mathcal{A}_l f$ is a C^∞ function on $\mathbf{R}$ with compact support. If $f, f_1, f_2 \in \mathcal{D}(G;\chi_l)$ and $a_1, a_2 \in \mathbf{C}$, then*

$$(\mathcal{A}_l f)^* = \mathcal{A}_l(f^*), \tag{3.17}$$

$$\mathcal{A}_l(a_1 f_1 + a_2 f_2) = a_1 \mathcal{A}_l f_1 + a_2 \mathcal{A}_l f_2, \tag{3.18}$$

$$\mathcal{A}_l(f_1 * f_2) = \mathcal{A}_l f_1 * \mathcal{A}_l f_2. \tag{3.19}$$

Formula (3.17) is equivalent to the fact that $\mathcal{A}_l f$ is an even function.

Proof. Formulas (3.17)–(3.19) are immediately proven by passing to $\mathcal{D}(G;\tau_l)$. For the last statement, recall that $f(g^{-1}) = f(g)$ for $f \in \mathcal{D}(G;\chi_l)$. □

The following lemma relates our definition of $\mathcal{A}_l$ to the definition often found in the literature (cf. e.g. [W2], p. 34).

3.3. LEMMA. *For $f \in \mathcal{D}(G;\chi_l)$ one has*

$$\int_N f(k a_t n)\, dn = \frac{1}{d_l}\chi_l(k) \int_N f(a_t n)\, dn.$$

Proof. Let $F \in \mathcal{D}(G;\tau_l)$. Then $\int_N F(a_t n)\, dn$ commutes with $\tau(m)$ ($m \in M$) so with $\tau(k_1)$ ($k_1 \in K_1$), hence is a scalar multiple of the identity. The lemma follows by taking traces. □

We now use the τ_l-Abel transform to construct τ_l-spherical functions. Because of Proposition 3.2, for any complex number s, the map

$$\lambda_s : f \longmapsto \int_{-\infty}^{\infty} \mathcal{A}_l f(t) e^{-st}\, dt \tag{3.20}$$

is an algebra homomorphism of $\mathcal{D}(G;\chi_l)$ into $\mathbf{C}$.

Set

$$\alpha_{l,s}(ka_tn) = \frac{1}{d_l}\chi_l(k)e^{-(s+\rho)t}. \tag{3.21}$$

Since $f = f * d_l\chi_l$ and $\chi_l(k^{-1}) = \chi_l(k)$ for $k \in K$, for every $f \in \mathcal{D}(G;\chi_l)$

$$\begin{aligned}\lambda_s(f) &= \frac{1}{d_l}\int_K\int_{-\infty}^{\infty}\int_N f(ka_tn)\chi_l(k)e^{(-s+\rho)t}\,dk\,dt\,dn\\ &= \int_G f(g)\alpha_{l,s}(g)\,dg\\ &= \int_G f(g)\int_K \alpha_{l,s}(kgk^{-1})\,dk\,dg\\ &= \int_G f(g)\zeta_{l,s}(g)\,dg \end{aligned} \tag{3.22}$$

with

$$\zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1})\,dk. \tag{3.23}$$

One easily checks that $\zeta_{l,s}$ satisfies $\zeta_{l,s} = \zeta_{l,s}{}^0$, $\zeta_{l,s}*d_l\chi_l = \zeta_{l,s}$ and $\zeta_{l,s}(e) = 1$. Thus $\zeta_{l,s}$ is a τ_l-spherical function. It will be shown in the next section that any τ_l-spherical function is of the form (3.24).

By Remark 2.3, we have

$$\zeta_{l,s}(g) = \frac{1}{d_l}\chi_l(k_1)\zeta_{l,s}(a_t) \quad \text{for } g = k_1k_2a_tk_2', \tag{3.24}$$

so $\zeta_{l,s}$ is uniquely determined by its restriction to A.

4. THE DIFFERENTIAL EQUATION FOR THE τ_l-SPHERICAL FUNCTIONS

For a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$, let $\mathfrak{u}_\mathbf{C}$ denote the complex subalgebra of $\mathfrak{g}_\mathbf{C}$ generated by $\mathfrak{u}$. The universal enveloping algebra $\mathfrak{U}(\mathfrak{u})$ of $\mathfrak{u}_\mathbf{C}$ is considered as a subalgebra of $\mathfrak{U}(\mathfrak{g})$.

The representation τ_l of K_1 induces differentiated representations of the Lie algebra $\mathfrak{k}_1$ of K_1 and of the universal enveloping algebra $\mathfrak{U}(\mathfrak{k}_1)$ of $(\mathfrak{k}_1)_\mathbf{C}$. We indicate these representations with the same letter τ_l. Let $\mathfrak{k}_2$ be the Lie algebra of K_2. Every element $Y \in \mathfrak{k}_\mathbf{C}$ can be uniquely decomposed as