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immersions. Given  $v \in V'$ , we need to show there exists  $x \in V'$  such that  $(x, v)$  and  $(v, x)$  are in  $Z'$ . This is true if  $v \in V$  by the property of  $Z$ . If  $v \in V_s$ , then  $v = a_s$  for some  $a \in V$ . We leave it to the reader to show that  $(x, a_s) \in Z_1$  and  $(a_s, x) \in Z_2$  for generic  $x$  in  $V$ . This completes the proof of the lemma.

The above lemma allows us to replace  $V$  by  $V'$ , hence to expand  $V$  whenever there exists a point  $s$  in  $V$  such that  $vs$  is not defined for all  $v \in V$ , and we can expand  $V'$  if there exists a point  $s' \in V'$  such that  $v's'$  is not defined for all  $v' \in V'$ . Denote the result of finitely many such expansions also by  $V'$ , and let  $U \subset V \times V \times V'$  be the closure of  $\Gamma$ . By Lemma 4.3 applied to  $V'$ , the projection  $p_{12}: U \rightarrow V \times V$  is an open immersion. Its image is the set of points  $(a, b)$  such that  $m: V \times V \rightarrow V'$  is defined at  $(a, b)$ . If  $V \times s \not\subset p_{12}(U)$  for some point  $s$  in  $V$ , then replacing  $V'$  by  $V' \cup V_s'$  increases both  $V'$  and  $p_{12}(U)$ . Using noetherian induction on open subschemes of  $V \times V$ , we may assume that after finitely many expansions,  $V \times s \subset p_{12}(U)$  for all points  $s \in V$ . Then we have  $p_{12}(U) = V \times V$ .

**PROPOSITION 4.5.** *Let  $V$ ,  $V'$ , and  $U$  be as above. If  $p_{12}(U) = V \times V$ , then the operation  $m: V' \times V' \rightarrow V'$  is everywhere defined on  $V'$  and makes  $V'$  an algebraic group.*

*Proof.* Take  $(a', b')$  in  $V' \times V'$ . Choose a point  $x$  so that  $a'x$  and  $x^{-1}b'$  are both defined and lie in  $V$ . Then we can define  $m(a', b') = (a'x)(x^{-1}b')$ . Similarly one can define  $a'^{-1}b'$  and  $b'a'^{-1}$ . In this way we extend  $m$ ,  $\Phi$ ,  $\Psi$ ,  $\Phi^{-1}$  and  $\Psi^{-1}$  to  $V' \times V'$ . The verification of the group axioms is routine and is omitted.

## 5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

Keep the notations in §3. We have proved that there is a birational group structure on  $(X - S)^{(\pi)}$ . The algebraic group associated to this birational group is called the *generalized jacobian* of  $X_m$  and is denoted by  $J_m$ . It is a commutative algebraic group.

Let  $D_0$  be a divisor on  $X$  prime to  $S$  of degree 0. By Lemma 3.3, the set

$$V_{D_0} = \{D \in (X - S)^{(\pi)} \mid l_m(D + D_0) = 1, \quad l(D + D_0 - m) = 0\}$$

is a non-empty open subset of  $(X - S)^{(\pi)}$ . We have the following

LEMMA 5.1. *There exists a unique morphism of varieties*

$$\alpha_{D_0}: V_{D_0} \rightarrow (X - S)^{(\pi)}$$

such that  $\alpha_{D_0}(D)$  is the unique effective divisor  $\mathfrak{m}$ -equivalent to  $D + D_0$  for any  $D \in V_{D_0}$ . Moreover  $\alpha_{D_0}$  is birational.

*Proof.* Consider the Cartesian squares

$$\begin{array}{ccccc} X_{\mathfrak{m}} \times V_{D_0} & \subset & X_{\mathfrak{m}} \times (X - S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ q \downarrow & & \downarrow & & \downarrow \\ V_{D_0} & \subset & (X - S)^{(\pi)} & \longrightarrow & \text{spec}(k). \end{array}$$

Let  $\mathcal{L}$  be the restriction to  $X_{\mathfrak{m}} \times V_{D_0}$  of the invertible sheaf on  $X_{\mathfrak{m}} \times (X - S)^{(\pi)}$  that corresponds to the divisor  $\mathcal{D} + p^*(D_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor. By Theorem 1.1(c) the sheaf  $q_*\mathcal{L}$  is invertible. The canonical map  $q^*q_*\mathcal{L} \rightarrow \mathcal{L}$  induces a homomorphism  $s: \mathcal{O}_{X_{\mathfrak{m}} \times V_{D_0}} \rightarrow \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$ . Using Remark 2.1, one can show that the pair  $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$  induces a relative effective Cartier divisor on  $(X_{\mathfrak{m}} \times V_{D_0})/V_{D_0}$ . Applying Proposition 3.1 to this divisor, one gets the existence of  $\alpha_{D_0}$ . For any  $D \in V_{D_0}$ , we have  $l_{\mathfrak{m}}(D + D_0) = 1$  and  $l(D + D_0 - \mathfrak{m}) = 0$ . So there is one and only one effective divisor  $\mathfrak{m}$ -equivalent to  $D + D_0$ , and this effective divisor is simply  $\alpha_{D_0}(D)$ .

We claim that  $\alpha_{-D_0}$  is the birational inverse of  $\alpha_{D_0}$ . We have

$$\begin{aligned} \alpha_{D_0}^{-1}(V_{-D_0}) &= \{D \mid D \in V_{D_0}, \alpha_{D_0}(D) \in V_{-D_0}\} \\ &= \{D \mid D \in V_{D_0}, l_{\mathfrak{m}}(\alpha_{D_0}(D) - D_0) = 1, l(\alpha_{D_0}(D) - D_0 - \mathfrak{m}) = 0\} \\ &= V_{D_0} \cap \{D \mid l_{\mathfrak{m}}(D) = 1, l(D - \mathfrak{m}) = 0\} \\ &= V_{D_0} \cap V_0. \end{aligned}$$

By Lemma 3.3 both  $V_{D_0}$  and  $V_0$  are open and non-empty. Since  $(X - S)^{(\pi)}$  is irreducible, the set  $V_{D_0} \cap V_0$  is also open and non-empty, that is,  $\alpha_{D_0}^{-1}(V_{-D_0})$  is open and non-empty. One can easily show that on this open set  $\alpha_{-D_0} \circ \alpha_{D_0}$  is defined and is the identity. Similarly one can show  $\alpha_{-D_0}^{-1}(V_{D_0})$  is open and non-empty, and on it  $\alpha_{D_0} \circ \alpha_{-D_0}$  is defined and is the identity. So  $\alpha_{D_0}$  is birational.

We have a birational map  $\varphi: (X - S)^{(\pi)} \rightarrow J_{\mathfrak{m}}$  by the construction of  $J_{\mathfrak{m}}$ . Let  $\text{dom}(\varphi)$  be an open subset of  $(X - S)^{(\pi)}$  such that  $\varphi|_{\text{dom}(\varphi)}$  is an open immersion. Moreover we may assume that for any  $a \in \text{dom}(\varphi)$ , both  $(a, x)$

and  $(x, a)$  lie in the set  $U$  defined in Lemma 3.4(a) if  $x$  is generic, i.e., lies in some open set. In particular,  $m(a, x)$  and  $m(x, a)$  are defined for generic  $x$ .

Let

$$U_{D_0} = V_{D_0} \cap \text{dom}(\varphi) \cap \alpha_{D_0}^{-1}(\text{dom}(\varphi)).$$

Note that  $U_{D_0}$  is open and non-empty since  $(X - S)^{(\pi)}$  is irreducible and  $\alpha_{D_0}$  is birational. Moreover  $\varphi(D)$  and  $\varphi(\alpha_{D_0}(D))$  are defined for any  $D \in U_{D_0}$ . Define

$$\theta_0(D_0) = \varphi(\alpha_{D_0}(D_0)) - \varphi(D_0).$$

LEMMA 5.2.  $\theta_0(D_0)$  does not depend on the choice of  $D$ .

*Proof.* Let  $D_1$  and  $D_2$  be two elements in  $U_{D_0}$ . We need to show that

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

Choose  $D_3 \in U_{D_0}$  so that  $(\alpha_{D_0}(D_1), D_3)$ ,  $(D_1, \alpha_{D_0}(D_3))$ ,  $(\alpha_{D_0}(D_2), D_3)$  and  $(D_2, \alpha_{D_0}(D_3))$  all lie in the set  $U$  defined in Lemma 3.4(a). Such a  $D_3$  exists. Indeed, if  $(\alpha_{D_0}(D_1), x)$ ,  $(D_1, x)$ ,  $(\alpha_{D_0}(D_2), x)$  and  $(D_2, x)$  all lie in  $U$  for  $x$  lying in an open set  $O$ , then we may choose  $D_3$  to be any element in  $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$ . Note that  $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$  is not empty since  $\alpha_{D_0}$  is birational and  $(X - S)^{(\pi)}$  is irreducible.

We have

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(m(\alpha_{D_0}(D_1), D_3)),$$

$$\varphi(D_1) + \varphi(\alpha_{D_0}(D_3)) = \varphi(m(D_1, \alpha_{D_0}(D_3))).$$

Since

$$m(\alpha_{D_0}(D_1), D_3) \sim_m \alpha_{D_0}(D_1) + D_3 - \pi P_0 \sim_m D_1 + D_0 + D_3 - \pi P_0,$$

$$m(D_1, \alpha_{D_0}(D_3)) \sim_m D_1 + \alpha_{D_0}(D_3) - \pi P_0 \sim_m D_1 + D_3 + D_0 - \pi P_0,$$

we have

$$m(\alpha_{D_0}(D_1), D_3) = m(D_1, \alpha_{D_0}(D_3)).$$

Hence

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(D_1) + \varphi(\alpha_{D_0}(D_3)),$$

that is,

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Similarly we have

$$\varphi(\alpha_{D_0}(D_2)) - \varphi(D_2) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Therefore

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

This proves the lemma.

Thus we have a well-defined map  $\theta_0: \text{Div}^{(0)} \rightarrow J_m$  from the set of divisors of degree 0 on  $X$  prime to  $S$  to  $J_m$ .

LEMMA 5.3.  $\theta_0$  is a homomorphism.

*Proof.* Let  $D_0, E_0 \in \text{Div}^{(0)}$  and let  $F_0 = D_0 + E_0$ . Choose  $D \in U_{D_0}$ ,  $E \in U_{E_0}$  and  $F \in U_{F_0}$  so that

$$(\alpha_{D_0}(D), \alpha_{E_0}(E)), (D, E), (m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F) \text{ and } (m(D, E), \alpha_{F_0}(F))$$

all lie in the set  $U$  defined in Lemma 3.4(a). We have

$$\begin{aligned} \alpha_{D_0}(D) + \alpha_{E_0}(E) + F &\sim_m D + D_0 + E + E_0 + F = D + E + F + D_0 + E_0, \\ D + E + \alpha_{F_0}(F) &\sim_m D + E + F + F_0 = D + E + F + D_0 + E_0. \end{aligned}$$

So

$$m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F) = m(m(D, E), \alpha_{F_0}(F)).$$

Hence

$$\varphi(m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F)) = \varphi(m(m(D, E), \alpha_{F_0}(F))).$$

Therefore

$$\varphi(\alpha_{D_0}(D)) + \varphi(\alpha_{E_0}(E)) + \varphi(F) = \varphi(D) + \varphi(E) + \varphi(\alpha_{F_0}(F)),$$

or equivalently,

$$(\varphi(\alpha_{D_0}(D)) - \varphi(D)) + (\varphi(\alpha_{E_0}(E)) - \varphi(E)) = \varphi(\alpha_{F_0}(F)) - \varphi(F).$$

This last equality is exactly

$$\theta_0(D_0) + \theta_0(E_0) = \theta_0(D_0 + E_0).$$

So  $\theta_0$  is a homomorphism.

We define  $\theta: \text{Div} \rightarrow J_m$  from the group of divisors on  $X$  prime to  $S$  to  $J_m$  by

$$\theta(D) = \theta_0(D - \deg(D)P_0).$$

Obviously  $\theta$  is a homomorphism.

PROPOSITION 5.4. *The homomorphism  $\theta$  is surjective and  $\ker(\theta)$  consists of divisors  $m$ -equivalent to integral multiples of  $P_0$ .*

*Proof.* Assume  $\sum_{i=1}^{\pi} P_i$  is in  $\text{dom}(\varphi)$ . We have

$$\theta\left(\sum_{i=1}^{\pi} P_i\right) = \theta_0\left(\sum_{i=1}^{\pi} P_i - \pi P_0\right) = \varphi(\alpha_{D_0}(D)) - \varphi(D),$$

where  $D_0 = \sum_{i=1}^{\pi} P_i - \pi P_0$  and  $D \in U_{D_0}$ . We may choose  $D$  so that  $m(\sum_{i=1}^{\pi} P_i, D)$  is defined and is the unique effective divisor  $m$ -equivalent to  $\sum_{i=1}^{\pi} P_i + D - \pi P_0$ . Since  $\alpha_{D_0}(D)$  is the unique effective divisor  $m$ -equivalent to  $D + D_0 = D + \sum_{i=1}^{\pi} P_i - \pi P_0$ , we have  $m(\sum_{i=1}^{\pi} P_i, D) = \alpha_{D_0}(D)$ . Hence  $\varphi(m(\sum_{i=1}^{\pi} P_i, D)) = \varphi(\alpha_{D_0}(D))$ . So  $\varphi(\sum_{i=1}^{\pi} P_i) + \varphi(D) = \varphi(\alpha_{D_0}(D))$ . Therefore  $\varphi(\alpha_{D_0}(D)) - \varphi(D) = \varphi(\sum_{i=1}^{\pi} P_i)$ , that is,

$$\theta\left(\sum_{i=1}^{\pi} P_i\right) = \varphi\left(\sum_{i=1}^{\pi} P_i\right).$$

This is true whenever  $\sum_{i=1}^{\pi} P_i$  is in  $\text{dom}(\varphi)$ .

Since  $\varphi|_{\text{dom}(\varphi)}$  is an open immersion,  $\varphi(\text{dom}(\varphi))$  is an open subset of  $J_m$ . The image of  $\theta$  contains this open subset. But  $J_m$  is generated by any open subset. So we must have  $\text{Im}(\theta) = J_m$  and  $\theta$  is surjective.

Assume  $E \in \ker(\theta)$ . Then  $\theta_0(E - \deg(E)P_0) = 0$ . Put  $E_0 = E - \deg(E)P_0$ . Then for any  $F \in U_{E_0}$ , we have

$$\varphi(\alpha_{E_0}(F)) - \varphi(F) = \theta_0(E - \deg(E)P_0) = 0.$$

Hence  $\varphi(\alpha_{E_0}(F)) = \varphi(F)$ . But  $\varphi$  is an open immersion on  $\text{dom}(\varphi)$ . So we have  $\alpha_{E_0}(F) = F$ . Since  $\alpha_{E_0}(F) \sim_m F + E_0$ , we have  $F \sim_m F + E_0$ . Hence  $E_0 \sim_m 0$ , that is,  $E \sim_m \deg(E)P_0$ . So  $E$  is  $m$ -equivalent to an integral multiple of  $P_0$ .

Conversely assume  $E$  is  $m$ -equivalent to an integral multiple of  $P_0$  and let us prove that  $\theta(E) = 0$ . Again let  $E_0 = E - \deg(E)P_0$ . Then  $E_0 \sim_m 0$ . Choose  $F \in U_{E_0} \cap U_0$ , where  $U_0$  is the set  $U_{D_0}$  defined before by taking  $D_0 = 0$ . We have

$$\begin{aligned} \theta(E) &= \theta_0(E_0) = \varphi(\alpha_{E_0}(F)) - \varphi(F), \\ \theta(0) &= \varphi(\alpha_0(F)) - \varphi(F). \end{aligned}$$

Note that  $F + E_0 \sim_m F$  since  $E_0 \sim_m 0$ . But  $\alpha_{E_0}(F)$  is the unique effective divisor  $m$ -equivalent to  $F + E_0$ , and  $\alpha_0(F)$  is the unique effective divisor  $m$ -equivalent to  $F$ . So we must have  $\alpha_{E_0}(F) = \alpha_0(F)$ . Therefore  $\theta(E) = \theta(0) = 0$ .

Regarding a point  $P$  in  $X - S$  as a divisor, we can calculate  $\theta(P)$ . In this way we get a map  $\theta: X - S \rightarrow J_m$ .

**PROPOSITION 5.5.** *The map  $\theta: X - S \rightarrow J_m$  is a morphism of algebraic varieties.*

*Proof.* Let  $P \in X - S$  and let  $D_0 = P - P_0$ . Fix a  $D \in U_{D_0}$ . Consider the set  $W_1 = \{R \in X - S \mid l_m(D + R - P_0) = 1\}$ . By the Riemann-Roch theorem, for any  $R$  in  $X - S$ , we have  $l_m(D + R - P_0) \geq 1$ . Applying Theorem 1.1 (b) to the projection  $q: X_m \times (X - S) \rightarrow X - S$  and the invertible sheaf corresponding to the divisor  $\mathcal{D} + p^*(D - P_0)$ , where  $\mathcal{D}$  is the universal relative effective Cartier divisor on  $X_m \times (X - S)$  and  $p: X_m \times (X - S) \rightarrow X_m$  is another projection, we see that  $W_1$  is open in  $X - S$ . Similarly one can show  $W_2 = \{R \in X - S \mid l(D + R - P_0 - m) = 0\}$  is also open in  $X - S$ . So  $W = W_1 \cap W_2 = \{R \in X - S \mid l_m(D + R - P_0) = 1, l(D + R - P_0 - m) = 0\}$  is open in  $X - S$ . It is non-empty since  $P \in W$  by our choice of  $D$ . By Proposition 3.1 we have a morphism  $\gamma: W \rightarrow (X - S)^{(\pi)}$  of algebraic varieties such that for every  $R \in W$ ,  $\gamma(R)$  is the unique effective divisor that is  $m$ -equivalent to  $D + R - P_0$ . Since  $\alpha_{R - P_0}(D)$  is the unique effective divisor that is  $m$ -equivalent to  $D + R - P_0$ , we have  $\gamma(R) = \alpha_{R - P_0}(D)$ . Replacing  $W$  by an open subset containing  $P$ , we may assume  $\text{Im}(\gamma) \subset \text{dom}(\varphi)$ . Note that for any  $R \in W$ , we have  $D \in U_{R - P_0}$ , and

$$\theta(R) = \theta_0(R - P_0) = \varphi((\alpha_{R - P_0}(D)) - \varphi(D) = \varphi(\gamma(R)) - \varphi(D),$$

that is,  $\theta(R) = \varphi(\gamma(R)) - \varphi(D)$ . So  $\theta = \varphi \circ \gamma - \varphi(D)$  on  $W$ . This proves  $\theta$  is a morphism of algebraic varieties in an open subset containing  $P$ . Since  $P \in X - S$  is arbitrary,  $\theta$  is a morphism of algebraic varieties.

The morphism  $\theta: X - S \rightarrow J_m$  induces a morphism of algebraic varieties  $\theta: (X - S)^{(\pi)} \rightarrow J_m$ .

**PROPOSITION 5.6.**  *$\theta: (X - S)^{(\pi)} \rightarrow J_m$  coincides with the birational map  $\varphi: (X - S)^{(\pi)} \rightarrow J_m$ . In particular  $\varphi$  is everywhere defined.*

*Proof.* Let  $\sum_{i=1}^{\pi} P_i \in \text{dom}(\varphi)$ . By the proof of Proposition 5.4, we have  $\varphi(\sum_{i=1}^{\pi} P_i) = \theta(\sum_{i=1}^{\pi} P_i)$ . So  $\varphi = \theta$  as rational maps.

Thus there is no difference between  $\varphi$  and  $\theta$ . From now on we denote the map  $\varphi$  also by  $\theta$ . We summarize what we have so far in the following theorem.

THEOREM 1. *There is a morphism of algebraic varieties  $\theta: X - S \rightarrow J_m$  satisfying the following properties:*

- (a) *The extension of  $\theta$  to the group of divisors on  $X$  prime to  $S$  induces, by passing to quotient, an isomorphism between the group  $C_m^0$  of classes of divisors of degree zero with respect to  $m$ -equivalence and the group  $J_m$ .*
- (b) *The extension of  $\theta$  to  $(X - S)^{(\pi)}$  induces a birational map from  $X^{(\pi)}$  to  $J_m$ .*

The following theorem characterizes  $J_m$  by a universal property:

THEOREM 2. *Let  $f: X \rightarrow G$  be a rational map from  $X$  to a commutative algebraic group  $G$  and assume  $m$  is a modulus for  $f$ . Then there is a unique homomorphism  $F: J_m \rightarrow G$  of algebraic groups such that  $f = F \circ \theta + f(P_0)$ .*

*Proof.* Replacing  $f$  by  $f - f(P_0)$ , we may assume  $f(P_0) = 0$ . Since  $m$  is a modulus for  $f$ , the extension of  $f$  to the group of divisors of  $X$  prime to  $S$  induces a homomorphism  $C_m^0 \rightarrow G$  by passing to quotient. By Theorem 1 (a) we have  $J_m \cong C_m^0$  as groups. So we have a homomorphism of groups  $F: J_m \rightarrow G$  such that  $f = F\theta$ . It remains to prove  $F$  is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map  $\theta: (X - S)^{(\pi)} \rightarrow J_m$ . Denote the extension of  $f$  to  $(X - S)^{(\pi)}$  by  $f'$ . Then  $F\theta = f'$ . Since  $\theta$  is birational, it induces an isomorphism between an open subvariety of  $(X - S)^{(\pi)}$  and an open subvariety of  $J_m$ . Moreover  $f'$  is a morphism of algebraic varieties. Hence  $F$  is a morphism of algebraic varieties when restricted to some open subset of  $J_m$ . The whole  $J_m$  can be obtained from this open subset by translation. So  $F$  is a morphism of algebraic varieties.

## 6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove  $J_m$  is the Picard scheme of  $X_m$ .

Let  $T$  be a  $k$ -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k) . \end{array}$$

We have  $q_*\mathcal{O}_{X_m \times T} = \mathcal{O}_T$  by [EGA] III, §1.4.15, the fact  $H^0(X_m, \mathcal{O}_{X_m}) = k$ , and the fact that  $T \rightarrow \text{spec}(k)$  is flat. The morphism  $q$  has a section  $s: T \rightarrow X_m \times T$ ,  $t \mapsto (P_0, t)$ .