6. Generalized jacobians and Picard schemes

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THEOREM 1. *There is a morphism of algebraic varieties $\theta\colon X - S \to J_{\mathfrak{m}}$ satisfying the following properties:*

(a) *The extension of θ to the group of divisors on X prime to S induces, by passing to quotient, an isomorphism between the group $C_{\mathfrak{m}}^0$ of classes of divisors of degree zero with respect to $\mathfrak{m}$-equivalence and the group $J_{\mathfrak{m}}$.*

(b) *The extension of θ to $(X-S)^{(\pi)}$ induces a birational map from $X^{(\pi)}$ to $J_{\mathfrak{m}}$.*

The following theorem characterizes $J_{\mathfrak{m}}$ by a universal property:

THEOREM 2. *Let $f\colon X \to G$ be a rational map from X to a commutative algebraic group G and assume $\mathfrak{m}$ is a modulus for f. Then there is a unique homomorphism $F\colon J_{\mathfrak{m}} \to G$ of algebraic groups such that $f = F \circ \theta + f(P_0)$.*

Proof. Replacing f by $f - f(P_0)$, we may assume $f(P_0) = 0$. Since $\mathfrak{m}$ is a modulus for f, the extension of f to the group of divisors of X prime to S induces a homomorphism $C_{\mathfrak{m}}^0 \to G$ by passing to quotient. By Theorem 1 (a) we have $J_{\mathfrak{m}} \cong C_{\mathfrak{m}}^0$ as groups. So we have a homomorphism of groups $F\colon J_{\mathfrak{m}} \to G$ such that $f = F\theta$. It remains to prove F is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map $\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$. Denote the extension of f to $(X-S)^{(\pi)}$ by f'. Then $F\theta = f'$. Since θ is birational, it induces an isomorphism between an open subvariety of $(X-S)^{(\pi)}$ and an open subvariety of $J_{\mathfrak{m}}$. Moreover f' is a morphism of algebraic varieties. Hence F is a morphism of algebraic varieties when restricted to some open subset of $J_{\mathfrak{m}}$. The whole $J_{\mathfrak{m}}$ can be obtained from this open subset by translation. So F is a morphism of algebraic varieties.

6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove $J_{\mathfrak{m}}$ is the Picard scheme of $X_{\mathfrak{m}}$.

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

We have $q_*\mathcal{O}_{X_{\mathfrak{m}}\times T} = \mathcal{O}_T$ by [EGA] III, §1.4.15, the fact $H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}) = k$, and the fact that $T \to \mathrm{spec}(k)$ is flat. The morphism q has a section $s\colon T \to X_{\mathfrak{m}} \times T$, $t \mapsto (P_0, t)$.

LEMMA 6.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on $X_{\mathfrak{m}} \times T$. Assume $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the canonical map* $\operatorname{Hom}(\mathcal{L}_1, \mathcal{L}_2) \to \operatorname{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_2)$ *induced by s is bijective.*

Proof. Since $\mathcal{L}_1 \cong \mathcal{L}_2$, it is enough to show that the canonical map $\operatorname{Hom}(\mathcal{L}_1, \mathcal{L}_1) \to \operatorname{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ is bijective. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_{\mathfrak{m}} \times T}(X_{\mathfrak{m}} \times T) & \longrightarrow & \mathcal{O}_T(T) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(\mathcal{L}_1, \mathcal{L}_1) & \longrightarrow & \operatorname{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)\,, \end{array}$$

where the horizontal arrows are induced by s. We have

$$\begin{aligned} \operatorname{Hom}(\mathcal{L}_1, \mathcal{L}_1) &\cong \operatorname{Hom}(\mathcal{O}_{X_{\mathfrak{m}} \times T}, \mathcal{L}_1 \otimes \mathcal{L}_1^{-1}) \\ &\cong \operatorname{Hom}(\mathcal{O}_{X_{\mathfrak{m}} \times T}, \mathcal{O}_{X_{\mathfrak{m}} \times T}) \cong \mathcal{O}_{X_{\mathfrak{m}} \times T}(X_{\mathfrak{m}} \times T)\,. \end{aligned}$$

Hence the left vertical arrow in the above diagram is bijective. Similarly the right vertical arrow is also bijective. Since $q_*\mathcal{O}_{X_{\mathfrak{m}} \times T} = \mathcal{O}_T$, we have $\mathcal{O}_{X_{\mathfrak{m}} \times T}(X_{\mathfrak{m}} \times T) \cong \mathcal{O}(T)$, and the upper horizontal arrow is bijective. Hence $\operatorname{Hom}(\mathcal{L}_1, \mathcal{L}_1) \cong \operatorname{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ by the commutativity of the above diagram.

LEMMA 6.2. *Let $\{U_i\}$ be an open covering of T and let $\mathcal{L}_i$ be invertible sheaves on $X_{\mathfrak{m}} \times U_i$. Assume $s^*\mathcal{L}_i \cong \mathcal{O}_{U_i}$ and $\mathcal{L}_i\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)} \cong \mathcal{L}_j\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}$. Then there exists an invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times T$ such that $\mathcal{L}\,|_{X_{\mathfrak{m}} \times U_i} \cong \mathcal{L}_i$ and $s^*\mathcal{L} \cong \mathcal{O}_T$. Moreover $\mathcal{L}$ is unique up to isomorphism.*

Proof. Fix an isomorphism $\alpha_i \colon s^*\mathcal{L}_i \to \mathcal{O}_{U_i}$ for each i. Let

$$\alpha_{ij} \colon s^*\mathcal{L}_i|_{U_i \cap U_j} \to s^*\mathcal{L}_j|_{U_i \cap U_j}$$

be the isomorphism $(\alpha_j|_{U_i \cap U_j})^{-1} \circ (\alpha_i|_{U_i \cap U_j})$. By Lemma 6.1 the canonical map

$$\operatorname{Hom}(\mathcal{L}_i|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}, \mathcal{L}_j|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}) \to \operatorname{Hom}(s^*\mathcal{L}_i|_{U_i \cap U_j}, s^*\mathcal{L}_j|_{U_i \cap U_j})$$

is bijective. So α_{ij} can be lifted uniquely to an isomorphism

$$A_{ij} \colon \mathcal{L}_i\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)} \to \mathcal{L}_j\,|_{X_{\mathfrak{m}} \times (U_i \cap U_j)}\,.$$

By the uniqueness of the lifting and the fact that $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ on $U_i \cap U_j \cap U_k$, we have $A_{jk}A_{ij} = A_{ik}$ on $X_{\mathfrak{m}} \times (U_i \cap U_j \cap U_k)$. So A_{ij} defines glueing data and we can glue the $\mathcal{L}_i$ together to get an invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times T$. By the construction of $\mathcal{L}$ we have $s^*\mathcal{L} \cong \mathcal{O}_T$. This proves the existence of $\mathcal{L}$. Similarly using Lemma 6.1 one can prove $\mathcal{L}$ is unique up to isomorphism.

LEMMA 6.3. *Assume T is integral. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on $X_{\mathrm{m}} \times T$ satisfying $\mathcal{L}_{1_t} \cong \mathcal{L}_{2_t}$ for all $t \in T$. Then there is an invertible sheaf $\mathcal{M}$ on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes q^*\mathcal{M}$.*

Proof. Let $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Then $\mathcal{L}_t \cong \mathcal{O}_{X_{\mathrm{m}}}$. It suffices to show that $\mathcal{L} \cong q^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on T. We have $H^0(X_{\mathrm{m}}, \mathcal{L}_t) = H^0(X_{\mathrm{m}}, \mathcal{O}_{X_{\mathrm{m}}}) = k$. By Theorem 1.1 (c), the sheaf $q_*\mathcal{L}$ is invertible and $q_*\mathcal{L} \otimes k(t) = H^0(X_{\mathrm{m}}, \mathcal{L}_t)$. So the restriction $(q^*q_*\mathcal{L})_t \to \mathcal{L}_t$ of the canonical map $q^*q_*\mathcal{L} \to \mathcal{L}$ to the fiber of q at $t \in T$ is $H^0(X_{\mathrm{m}}, \mathcal{L}_t) \otimes \mathcal{O}_{X_{\mathrm{m}}} \to \mathcal{L}_t$, which is an isomorphism since $\mathcal{L}_t \cong \mathcal{O}_{X_{\mathrm{m}}}$. By Nakayama's Lemma, the canonical map $q^*q_*\mathcal{L} \to L$ is surjective. But since it is a homomorphism of invertible sheaves, it must be bijective. Hence $\mathcal{L} \cong q^*q_*\mathcal{L}$.

Now we use the above lemmas to construct a canonical invertible sheaf on $X_{\mathrm{m}} \times J_{\mathrm{m}}$.

On $X_{\mathrm{m}} \times (X-S)^{(\pi)}$ we have the invertible sheaf corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor and $p\colon X_{\mathrm{m}} \times (X-S)^{(\pi)} \to X_{\mathrm{m}}$ is the projection. Since $\theta\colon (X-S)^{(\pi)} \to J_{\mathrm{m}}$ is birational, there exist open subsets U in $(X-S)^{(\pi)}$ and V in J_{m} such that θ induces an isomorphism $U \cong V$. Hence we can push-forward the above invertible sheaf on $X_{\mathrm{m}} \times (X-S)^{(\pi)}$ to get an invertible sheaf $\mathcal{L}_V$ on $X_{\mathrm{m}} \times V$. For each $t \in J_{\mathrm{m}}$, denote by $\mathcal{L}(t)$ the invertible sheaf on X_{m} corresponding to the divisor class in C^0_{m} that is mapped to $t \in J_{\mathrm{m}}$ under the canonical isomorphism $C^0_{\mathrm{m}} \cong J_{\mathrm{m}}$. Obviously the restriction $\mathcal{L}_{V,t}$ of $\mathcal{L}_V$ to the fiber of the projection $q\colon X_{\mathrm{m}} \times J_{\mathrm{m}} \to J_{\mathrm{m}}$ at $t \in V$ is isomorphic to $\mathcal{L}(t)$. The invertible sheaf $\mathcal{L}_V \otimes (q^*s^*\mathcal{L}_V)^{-1}$ has the same property, where $s\colon J_{\mathrm{m}} \to X_{\mathrm{m}} \times J_{\mathrm{m}}$ is the section $t \mapsto (P_0, t)$. Thus replacing $\mathcal{L}_V$ by $\mathcal{L}_V \otimes (q^*s^*\mathcal{L}_V)^{-1}$ if necessary, we may assume that $s^*\mathcal{L}_V \cong \mathcal{O}_V$.

For each $a \in J_{\mathrm{m}}$, let $T_{-a}\colon J_{\mathrm{m}} \to J_{\mathrm{m}}$ be the translation $t \mapsto t-a$. Consider the invertible sheaf $\mathcal{L}_{a+V} = (\mathrm{id} \times T_{-a})^*\mathcal{L}_V \otimes p^*\mathcal{L}(a)$ on $X_{\mathrm{m}} \otimes (a+V)$, where $p\colon X_{\mathrm{m}} \times J_{\mathrm{m}} \to X_{\mathrm{m}}$ is the projection. The restriction $\mathcal{L}_{a+V,a+t}$ of $\mathcal{L}_{a+V}$ to the fiber of q at $a+t \in a+V$ is

$$((\mathrm{id} \times T_{-a})^*\mathcal{L}_V \otimes p^*\mathcal{L}(a))_{a+t} = \mathcal{L}_{V,t} \otimes \mathcal{L}(a) = \mathcal{L}(t) \otimes \mathcal{L}(a) = \mathcal{L}(a+t),$$

that is, $\mathcal{L}_{a+V,a+t} = \mathcal{L}(a+t)$. Hence for any $t \in V \cap (a+V)$, we have $\mathcal{L}_{V,t} = \mathcal{L}_{a+V,t}$. By Lemma 6.3, we have

$$\mathcal{L}_V|_{X_{\mathrm{m}} \times (V \cap (a+V))} \cong \mathcal{L}_{a+V}|_{X_{\mathrm{m}} \times (V \cap (a+V))} \otimes q^*\mathcal{M}$$

for some invertible sheaf $\mathcal{M}$ on $V \cap (a+V)$. But since $s^*\mathcal{L}_V \cong \mathcal{O}_V$, we also have $s^*\mathcal{L}_{a+V} = \mathcal{O}_{a+V}$. Hence $\mathcal{M} \cong \mathcal{O}_{V \cap (a+V)}$. Therefore $\mathcal{L}_V|_{X_{\mathrm{m}} \times (V \cap (a+V))} \cong$

$\mathcal{L}_{a+V}|_{X_{\mathfrak{m}}\times(V\cap(a+V))}$. By Lemma 6.2, we can glue $\mathcal{L}_{a+V}$ $(a \in J_m)$ together to get an invertible sheaf $\mathcal{L}_{J_{\mathfrak{m}}}$ on $X_{\mathfrak{m}} \times J_{\mathfrak{m}}$. It has the property that its restriction to the fiber of q at $t \in J_{\mathfrak{m}}$ is isomorphic to $\mathcal{L}(t)$ and $s^*\mathcal{L}_{J_{\mathfrak{m}}} \cong \mathcal{O}_{J_{\mathfrak{m}}}$.

Define

$$P^0(T) = \{\mathcal{L} \in \mathrm{Pic}(X_{\mathfrak{m}} \times T) \mid \deg(\mathcal{L}) = 0\}/q^*\,\mathrm{Pic}(T),$$

where $\deg(\mathcal{L})$ is defined as the leading coefficient of $\chi(\mathcal{L}_t^{\otimes n})$ as a polynomial in n. Since $s^*q^* = \mathrm{id}$, we may define

$$P^0(T) = \{\mathcal{L} \in \mathrm{Pic}(X_{\mathfrak{m}} \times T) \mid \deg(\mathcal{L}) = 0 \text{ and } s^*\mathcal{L} \cong \mathcal{O}_T\}$$

as well. In particular, we have $\mathcal{L}_{J_{\mathfrak{m}}} \in P^0(J_{\mathfrak{m}})$. Using the first definition of $P^0(T)$ and Lemma 6.3, one can show that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times \theta\colon X_{\mathfrak{m}} \times (X-S)^{(\pi)} \to X_{\mathfrak{m}} \times J_{\mathfrak{m}}$ is the invertible sheaf on $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$.

The following theorem says that $J_{\mathfrak{m}}$ is the Picard scheme of $X_{\mathfrak{m}}$.

THEOREM 3. *The functor $T \to P^0(T)$ is represented by $J_{\mathfrak{m}}$. More precisely, for any invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times T$ of degree 0 satisfying $s^*\mathcal{L} \cong \mathcal{O}_T$, there is one and only one morphism of schemes $f\colon T \to J_{\mathfrak{m}}$ such that $\mathcal{L}$ is the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f\colon X_{\mathfrak{m}} \times T \to X_{\mathfrak{m}} \times J_{\mathfrak{m}}$.*

Proof. Let $V_0 = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D) = 1, \quad l(D-\mathfrak{m}) = 0\}$. By Lemma 3.3, we know V_0 is non-empty and open in $(X-S)^{(\pi)}$. Note that for every $D \in V_0$, there is one and only one effective divisor in $X_{\mathfrak{m}}$ that is $\mathfrak{m}$-equivalent to D. Hence the restriction $\theta|_{V_0}$ of $\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ to V_0 is injective. By [EGA] III, §4.4.9, $\theta|_{V_0}$ is an open immersion.

Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \xrightarrow{p} & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

Let $\mathcal{L}' = \mathcal{L} \otimes p^*\mathcal{L}(\pi P_0)$, where $\mathcal{L}(\pi P_0)$ is the invertible sheaf on $X_{\mathfrak{m}}$ corresponding to the divisor πP_0. Let us prove the theorem under the extra assumption that for every $t \in T$, we have $\dim H^0(X_{\mathfrak{m}}, \mathcal{L}'_t) = 1$ and $\dim H^0(X, \mathcal{L}'_t \otimes \mathcal{L}(-\mathfrak{m})) = 0$, where $\mathcal{L}(-\mathfrak{m})$ is the invertible sheaf on X corresponding to the divisor $-\mathfrak{m}$. By the Riemann-Roch theorem, for every $t \in T$, we have $\dim H^1(X_{\mathfrak{m}}, \mathcal{L}'_t) = 0$. By Theorem 1.1 (d) the sheaf $q_*\mathcal{L}'$ is invertible. The canonical map $q^*q_*\mathcal{L}' \to \mathcal{L}'$ induces

$$s\colon \mathcal{O}_{X_{\mathfrak{m}}\times T} \longrightarrow \mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}\,.$$

Using Remark 2.1, one can show that the pair $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$ defines a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T)/T$. By Proposition 3.1, there exists a unique morphism of schemes $g\colon T \longrightarrow (X - S)^{(\pi)}$ such that the pull-back by $\mathrm{id} \times g$ of the universal relative effective Cartier divisor $\mathcal{D}$ is the divisor defined by $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$. Let $f = \theta g$. Then the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f$ is $\mathcal{L}$. This proves the existence of f. To prove f is unique, assume $f\colon T \longrightarrow J_{\mathfrak{m}}$ is a morphism such that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f$ is $\mathcal{L}$. By our extra assumption, we must have $\mathrm{Im}(f) \subset \theta(V_0)$. But $\theta|_{V_0}$ is an open immersion. So there exists a morphism $g\colon T \longrightarrow (X - S)^{(\pi)}$ such that $f = \theta g$. We leave it to the reader to prove that the pull-back of the universal relative effective Cartier divisor $\mathcal{D}$ by $\mathrm{id} \times g$ is the divisor defined by the pair $(\mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1}, s)$. By Proposition 3.1, such kind of g is unique. So f is also unique.

Now let us prove the theorem. Let t_0 be a point in T. For every point $D \in (X - S)^{(\pi)}$, denote by $\mathcal{L}(D)$ the invertible sheaf on X or on $X_{\mathfrak{m}}$ corresponding to the divisor D. By Lemma 3.3, the set

$$\{D \in (X - S)^{(\pi)} \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{t_0} \otimes \mathcal{L}(D)) = 1,\ \dim H^0(X, \mathcal{L}_{t_0} \otimes \mathcal{L}(D - \mathfrak{m})) = 0\}$$

is non-empty (and open). Fix an element D in this set. Consider the set

$$U_{t_0} = \{t \in T \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t \otimes \mathcal{L}(D)) = 1,\ \dim H^0(X, \mathcal{L}_t \otimes \mathcal{L}(D - \mathfrak{m})) = 0\}\,.$$

This set is open by the Riemann-Roch theorem and Theorem 1.1 (b). Obviously it contains t_0. So U_{t_0} is an open neighbourhood of t_0. By the theorem with the extra assumption that we have already proved, there exists a unique morphism $f'_{U_{t_0}}\colon U_{t_0} \longrightarrow J_{\mathfrak{m}}$ such that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f'_{U_{t_0}}$ is $(\mathcal{L} \otimes p^* \mathcal{L}(D - \pi P_0))|_{X_{\mathfrak{m}} \times U_{t_0}}$. Put $f_{U_{t_0}} = f'_{U_{t_0}} + a$, where a is the point in $J_{\mathfrak{m}}$ corresponding to the divisor class $\pi P_0 - D$ in $C^0_{\mathfrak{m}}$. Obviously the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by the morphism $\mathrm{id} \times f_{U_{t_0}}$ is $\mathcal{L}\,|_{X_{\mathfrak{m}} \times U_{t_0}}$. Moreover, such an $f_{U_{t_0}}$ is unique. So we can glue $f_{U_{t_0}}$ together to get $f\colon T \longrightarrow J_{\mathfrak{m}}$.