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A GENERALIZED FØLNER CONDITION AND THE NORMS OF RANDOM WALK OPERATORS ON GROUPS

by Andrzej ŻUK

ABSTRACT. We prove a generalized Følner condition. We present a method of computing and estimating the norms of random walk operators on groups and graphs. We give explicit computations in several cases.

1. INTRODUCTION

Let us consider a pair (Γ, S), where Γ is a finitely generated group and S is a finite, symmetric set of generators (symmetric means $S = S^{-1}$).

For a finite subset $A \subset \Gamma$ we define its *boundary*

$$\partial A = \{\gamma \in A; \text{ there exists } s \in S \text{ such that } \gamma s \notin A\}.$$

A *Følner sequence* is a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of Γ such that the cardinality of the boundary ∂A_n of the set A_n divided by the cardinality of A_n tends to zero, i.e.

$$\frac{\#\partial A_n}{\#A_n} \to_{n\to\infty} 0.$$

Følner proved in [4] that the existence of such a sequence is equivalent to amenability of the group Γ.

One can associate with the pair (Γ, S) the *simple random walk operator* $P: l^2(\Gamma) \to l^2(\Gamma)$:

$$Pf(\gamma) = \frac{1}{\#S} \sum_{s\in S} f(\gamma s) \quad \text{for } f \in l^2(\Gamma).$$

Let $||P||$ be the operator norm of P acting on $l^2(\Gamma)$. In [8] Kesten proved:

THEOREM 1 (Kesten). *The following conditions are equivalent:*

(1) $||P|| = 1$.

(2) *The group* Γ *is amenable, i.e. there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* Γ *satisfying the Følner condition.*

In the next section we will prove a generalization of this result (Theorems 2 and 3), showing that equalities of the form $||P|| = \lambda$, with $0 < \lambda \leq 1$, are equivalent to appropriate Følner-like conditions. Section 3 is devoted to some remarks concerning this generalization. In Section 4 we use the generalized Følner condition to compute the norms of some random walk operators and in Section 5, using the same ideas, we obtain some lower bounds for the random walk operators on graphs.

After completion of this work, we learned that some versions of a generalized Følner condition were obtained recently by S. Popa [12].

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2. THE GENERALIZED FØLNER CONDITION

Let us consider a measurable space $(X, \mathcal{F})$. On this space we consider a *Markov transition kernel* $P(\cdot, \cdot)$, i.e. for any $x \in X$, $P(x, \cdot)$ is a probability measure on $(X, \mathcal{F})$ and $P(\cdot, A)$ is a measurable function on $(X, \mathcal{F})$ for every $A \in \mathcal{F}$.

Let μ be a σ-finite measure on the space $(X, \mathcal{F})$. For any measurable subset $A \subset X$ we define its measure $|A|$ and the measure $|\partial A|$ of its boundary ∂A as follows:

$$|A| = \mu(A),$$

$$|\partial A| = \int_{\{x \in A\}} \int_{\{y \in A^c\}} P(x, dy) d\mu(x).$$

We will suppose that the measure

$$dm(x,y) = d\mu(x)P(x,dy) \tag{1}$$

is symmetric on $X \times X$. Let P be the *Markov operator* acting on $L^2(X,\mu)$ as

$$Pf(x) = \int_{\{y\in X\}} f(y)P(x,dy)\,.$$

The above equation defines also an operator on the space of positive measurable functions on X.

When condition (1) is satisfied we say that P is *reversible* with respect to μ. The Markov operator P is a self-adjoint operator on $L^2(X,\mu)$ if and only if P is reversible with respect to μ.

We denote by $\langle\cdot,\cdot\rangle_{L^2(X,\mu)}$ the scalar product on $L^2(X,\mu)$ and by $\|P\|$ the norm of P acting on $L^2(X,\mu)$.

For a real-valued measurable function f and for a measurable subset $A \subset X$ let us define a relative measure $|A|_{f^2}$ and a relative measure of its boundary $|\partial A|_{f^2}$:

$$|A|_{f^2} = \int_{\{x\in A\}} f^2 d\mu(x)\,,$$
$$|\partial A|_{f^2} = \int_{\{x\in A\}}\int_{\{y\in A^c\}} f(x)f(y)P(x,dy)d\mu(x)\,.$$

THEOREM 2. *Let P be a Markov operator on a measurable space $(X,\mathcal{F})$, which is reversible with respect to a measure μ. Let f be a positive eigenfunction of P with a positive eigenvalue λ. Then the following conditions are equivalent:*

(1) *There is a constant $c > 0$ such that for any measurable subset $A \subset X$ of finite measure*

$$|A|_{f^2} \le c|\partial A|_{f^2}\,,$$

(2) $\|P\| < \lambda$.

In the case where X is a Cayley graph of a group Γ with a finite set of generators $S = S^{-1}$ like in Part 1, one can give the following formulation of the above theorem.

THEOREM 3. *Let f be a positive eigenfunction for the simple random walk operator P on the group Γ generated by a finite symmetric set S, with the eigenvalue λ, i.e.*

$$Pf = \lambda f .$$

The following conditions are equivalent:

(1) $\|P\| = \lambda$.

(2) (Generalized Følner condition) *There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of Γ, such that*

$$\frac{\sum_{\gamma \in \partial A_n} f^2(\gamma)}{\sum_{\gamma \in A_n} f^2(\gamma)} \rightarrow_{n\rightarrow\infty} 0 .$$

REMARK. In case where $\lambda = 1$ we can take the function f of Theorem 3 to be a constant function. We then obtain Kesten's theorem (Theorem 1).

There are also examples of amenable groups (see [2]) for which there exist eigenfunctions of the simple random walk operator corresponding to the eigenvalue equal to one and which are not constant. The generalized Følner condition applies also to them.

Theorem 2 will be deduced from the following proposition.

PROPOSITION 1 ([7,13]). *Let Q be a Markov operator on $(X, \mathcal{F})$ which is reversible with respect to a measure μ. Assume that there exists a constant $c > 0$ such that for any measurable subset $A \subset X$ of finite measure*

$$|A| \leq c|\partial A| . \tag{2}$$

Then

$$\|Q\|_{L^2(X,\mu)\rightarrow L^2(X,\mu)} \leq 1 - \frac{1}{\sqrt{2}c} < 1 .$$

In order to give a clear proof of Proposition 1, we need the following lemma.

LEMMA 1 ([13]). *For a non-negative measurable function f with compact support in X one has*

$$\int_{\{x\in X\}} \int_{\{y\in X\}} |f(x) - f(y)|Q(x, dy)d\mu(x) = 2\int_0^{\infty} |\partial\{f > t\}|dt .$$

Proof.

$$\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)|Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\int_{\{y\in X;f(x)>f(y)\}}(f(x)-f(y))Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\int_{\{y\in X;f(x)>f(y)\}}\int_0^\infty \mathbf{1}_{[f(y),f(x))}(t)dtQ(x,dy)d\mu(x)$$
$$=2\int_0^\infty\int_{\{x\in X\}}\int_{\{y\in X;f(x)>f(y)\}}\mathbf{1}_{[f(y),f(x))}(t)Q(x,dy)d\mu(x)dt$$
$$=2\int_0^\infty\Big(\int_{\{x\in X;f(x)>t\}}\int_{\{y\in X;f(y)\le t\}}Q(x,dy)d\mu(x)\Big)dt$$
$$=2\int_0^\infty|\partial\{f>t\}|dt\,.\quad\square$$

Proof of Proposition 1. Let us consider a real-valued measurable function f with compact support in X. The above lemma applied to the function f^2 and the strong isoperimetry condition (2) gives:

$$\int_{\{x\in X\}}\int_{\{y\in X\}}\left|f^2(x)-f^2(y)\right|Q(x,dy)d\mu(x)=2\int_0^\infty\left|\partial\{f^2>t\}\right|dt$$
$$\geq\frac{2}{c}\int_0^\infty\left|\{f^2>t\}\right|dt=\frac{2}{c}\int_X f^2(x)d\mu(x)\,.$$

On the other hand

$$\int_{\{x\in X\}}\int_{\{y\in X\}}\left|f^2(x)-f^2(y)\right|Q(x,dy)d\mu(x)$$
$$\le\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)|(|f(x)|+|f(y)|)Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)||f(x)|Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\Big(\int_{\{y\in X\}}|f(x)-f(y)|Q(x,dy)\Big)|f(x)|d\mu(x)$$
$$\le 2\int_{\{x\in X\}}\Big(\int_{\{y\in X\}}|f(x)-f(y)|^2Q(x,dy)\Big)^{\frac{1}{2}}|f(x)|d\mu(x)$$
$$\le 2\Big(\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)|^2Q(x,dy)d\mu(x)\Big)^{\frac{1}{2}}\Big(\int_{\{x\in X\}}|f(x)|^2d\mu(x)\Big)^{\frac{1}{2}}$$
$$=2\sqrt{2}\langle(I-Q)f,f\rangle_{L^2(X,\mu)}^{\frac{1}{2}}\|f\|_{L^2(X,\mu)}\,.$$

Hence

$$\langle (I-Q)f,f\rangle_{L^2(X,\mu)} \geq \frac{1}{\sqrt{2}c}\|f\|_{L^2(X,\mu)}\,.$$

As Q is a self-adjoint operator, this is equivalent to

$$\|Q\|_{L^2(X,\mu)\to L^2(X,\mu)} \leq 1-\frac{1}{\sqrt{2}c} < 1\,. \qquad \square$$

Let P be a Markov operator, reversible with respect to the measure μ. Let f be a positive eigenfunction of P for the eigenvalue λ.

LEMMA 2. *The operator defined by the kernel*

$$Q(x,dy) = \lambda^{-1}f(x)^{-1}P(x,dy)f(y)$$

is a Markov operator and is reversible with respect to the measure $f^2\mu$.

Proof. The kernel $Q(x,dy)$ is a Markov transition kernel, because:

$$\int_{\{y\in X\}} Q(x,dy) = \lambda^{-1}f(x)^{-1}\int_{\{y\in X\}} P(x,dy)f(y) = \lambda^{-1}f(x)^{-1}\lambda f(x) = 1\,.$$

In order to prove reversibility of Q, we have to prove that the measure

$$dm'(x,y) = f^2(x)d\mu(x)Q(x,dy)$$

is symmetric on $X\times X$, knowing that the measure

$$dm(x,y) = d\mu(x)P(x,dy)$$

is symmetric on $X\times X$.

This follows from the following equalities, where B is a measurable subset of $X\times X$:

$$\begin{aligned}\int_B dm'(x,y) &= \int_B f^2(x)d\mu(x)Q(x,dy)\\ &= \lambda^{-1}\int_B f(x)d\mu(x)P(x,dy)f(y)\\ &= \lambda^{-1}\int_B f(x)f(y)dm(x,y)\\ &= \lambda^{-1}\int_B f(x)f(y)dm(y,x)\\ &= \int_B dm'(y,x)\,. \qquad \square\end{aligned}$$

Proof of Theorem 2. Clearly, condition (2) in Theorem 2 implies condition (1). In order to prove the converse let us consider the Markov operator Q defined in the previous lemma and the measure $f^2\mu$ on X. Here we add to the notation for $|A|$ and $|\partial A|$ an index $(Q,f^2\mu)$ in order to distinguish when these notions are used for (P,μ) or for $(Q,f^2\mu)$. One has

$$\begin{aligned} |A|^{(Q,f^2\mu)} &= \int_{\{x\in A\}} f^2(x)d\mu(x) = |A|_{f^2} \\ |\partial A|^{(Q,f^2\mu)} &= \int_{\{x\in A\}}\int_{\{y\in A^c\}} \frac{1}{\lambda} f^{-1}(x)P(x,dy)f(y)f^2(x)d\mu(x) \\ &= \frac{1}{\lambda}\int_{\{x\in A\}}\int_{\{y\in A^c\}} f(x)f(y)P(x,dy)d\mu(x) = \frac{1}{\lambda}|\partial A|_{f^2}\,. \end{aligned}$$

The first condition implies that there exists $c' > 0$ such that

$$c'|\partial A|^{(Q,f^2\mu)} \geq |A|^{(Q,f^2\mu)},$$

which by Proposition 1 implies that

$$\|Q\|_{L^2(X,f^2\mu)\to L^2(X,f^2\mu)} < 1\,.$$

Let $\rho = \|Q\|_{L^2(X,f^2\mu)\to L^2(X,f^2\mu)}$. For any $g \in L^2(X,\mu)$:

$$\begin{aligned} \langle Pg, g\rangle_{L^2(X,\mu)} &= \lambda \left\langle Q\left(\frac{g}{f}\right), \frac{g}{f}\right\rangle_{L^2(X,f^2\mu)} \leq \lambda\rho\left\langle \frac{g}{f}, \frac{g}{f}\right\rangle_{L^2(X,f^2\mu)} \\ &= \lambda\rho\,\langle g, g\rangle_{L^2(X,\mu)}\,. \end{aligned}$$

As P is a self-adjoint operator and $\rho < 1$, this implies

$$\|P\|_{L^2(X,\mu)\to L^2(X,\mu)} < \lambda\,. \qquad \square$$

Proof of Theorem 3. One knows (see Section 3) that P has positive eigenfunctions only for the eigenvalues greater than or equal to $\|P\|$. So the second condition implies the first one.

In order to prove the converse, we remark that for $\gamma \sim \gamma' \in \Gamma$ one has:

$$\frac{1}{\lambda(\#S)} f(\gamma') \leq f(\gamma) \leq \lambda(\#S) f(\gamma'),$$

$$P(\gamma,\gamma') = \frac{1}{\#S}\,.$$

This implies that

$$\frac{1}{\#S}|A|_{f^2} = \sum_{\gamma\in A} f^2(\gamma)\,,$$

$$\frac{1}{\lambda(\#S)^2}|\partial A|_{f^2} \leq \sum_{\gamma\in\partial A} f^2(\gamma) \leq \lambda|\partial A|_{f^2}\,.$$

By Theorem 2, the first condition implies the second one. □

REMARK. The proof of Theorem 3 can easily be generalized to the case where P is a convolution operator with a finitely supported probability measure.

3. REMARKS

We will now make some comments about Theorems 2 and 3. We will state some theorems about the existence of eigenfunctions for the Markov operator and discuss whether one can take in the generalized Følner condition the eigenfunctions to be in $L^2(X,\mu)$.

For simplicity we will suppose that X is a connected, locally finite graph (i.e. the degree of each vertex is finite) and we consider the *simple random walk* going with equal probability from one vertex to any of its neighbors. We associate with this random walk the simple random walk operator P defined by

$$Pf(v) = \frac{1}{N(v)}\sum_{w\sim v} f(w) \quad \text{for } f\in l^2(X,N)$$

where $N(v)$ is the degree of vertex v in X (*i.e.* the number of edges adjacent to v), where $w\sim v$ means that w and v are connected by an edge and where $l^2(X,N)$ is the space of real-valued functions f on the vertices of X such that $\sum_{x\in X} f^2(x)N(x)$ is finite.

3.1 EXISTENCE OF EIGENFUNCTIONS

THEOREM 4 ([20]). *Let X be an infinite, locally finite graph and let P be the simple random walk operator on $l^2(X,N)$. For any $\lambda \geq \|P\|$ there exists a positive eigenfunction f of P with eigenvalue λ, i.e.*

$$Pf(x) = \lambda f(x) \quad \textit{and } f(x) > 0 \textit{ for } x\in X\,.$$

For $\lambda < \|P\|$ there are no positive eigenfunctions of P with eigenvalue λ.

Proof. There are several proofs of this theorem. In [20] one can find the proof where the analogue of Perron-Frobenius theory is developed and in [11] the truncation method is used. □

3.2 Eigenfunctions in l^2

One can ask whether the positive eigenfunctions of the random walk operator are in $l^2(X,N)$. The answer is no in the case when X is the Cayley graph of an infinite group Γ (see Theorem 5). But in the general case there are examples of eigenfunctions which are in $l^2(X,N)$ (see Proposition 2).

3.2.1 The case of groups

THEOREM 5. *Let f be a positive eigenfunction of the simple random walk operator P on the group Γ generated by a finite symmetric set S, i.e. $Pf = \lambda f$. If Γ is infinite then*

$$\sum_{\gamma\in\Gamma} f^2(\gamma) = +\infty .$$

Proof. Suppose the contrary, i.e. that there is a positive eigenfunction f of the operator P for which the l^2 norm is finite:

$$Pf_0 = \lambda f_0 ,$$
$$\sum_{\gamma\in\Gamma} f_0^2(\gamma) < +\infty .$$

The second condition implies that f_0 is not constant and so there are $\gamma_0, \gamma_1 \in \Gamma$ such that

$$f_0(\gamma_0) < f_0(\gamma_1) .$$

Let us define the function f_1 as a translation of f_0 by $\gamma_0\gamma_1^{-1}$, i.e.

$$f_1(\gamma) = f_0(\gamma_0\gamma_1^{-1}\gamma) .$$

The function f_1, being the translation of f_0, is an eigenfunction of P, i.e.

$$Pf_1 = \lambda f_1 .$$

So the function $\widetilde{f}$ defined as follows:

$$\widetilde{f}(\gamma) = \max\{f_0(\gamma), f_1(\gamma)\} ,$$

satisfies

$$P\widetilde{f} \geq \lambda \widetilde{f} .$$

As f_0 and f_1 are in $l^2(\Gamma)$, the function $\widetilde{f}$ is in $l^2(\Gamma)$ as well. The functions f_0 and f_1 have the same l^2 norms and

$$f_1(\gamma_1) = f_0(\gamma_0) < f_0(\gamma_1),$$

so there exists $\gamma_2 \in \Gamma$ such that

$$f_1(\gamma_2) > f_0(\gamma_2).$$

Note that these two inequalities imply that $\widetilde{f} \geq f_0$ with equality at some points and strict inequality at some other points. Thus $g = \widetilde{f} - f_0$ satisfies $g \geq 0$, $g \neq 0$, g vanishes at some points and $Pg \geq \lambda g$. Let us prove that this implies $P\widetilde{f} \neq \lambda \widetilde{f}$. Indeed, if we had equality then $Pg = \lambda g$ as well and thus $P^n g = \lambda^n g$. Taking n large enough makes $P^n g$ non-zero at points where g vanishes, a contradiction. We have thus shown that $P\widetilde{f} \geq \lambda \widetilde{f}$ with $P\widetilde{f} \neq \lambda \widetilde{f}$.

This means that

$$\|P\widetilde{f}\|_{l^2(\Gamma)} > \lambda \|\widetilde{f}\|_{l^2(\Gamma)}.$$

Hence

$$\|P\| > \lambda.$$

But this provides the desired contradiction because by Theorem 4 there are no positive eigenfunctions of P with an eigenvalue smaller than the norm of P. □

3.2.2 The general case

It will be shown that there are examples of the infinite graph X and the simple random walk operator P for which there is a positive eigenfunction in $l^2(X,N)$. It was pointed out to us by the referee that when P is the adjacency operator, examples of infinite graphs with positive eigenvalues in l^2 can be found for instance in [5] (page 232).

Let X be a uniform tree (*i.e.* a simply connected graph) of degree 3. By a theorem of Kesten (see [9]) one knows that $\|P\| = \frac{2}{3}\sqrt{2} < 1$. Let a and b be two neighboring vertices in X. Now let X_n be a graph which is the same as the graph X, except that the edge (a,b) is subdivided into n vertices. Let I_n denote the set of vertices a, b and added vertices which we label $1,\ldots,n$ (see Figure 1). Let P_n be the simple random walk operator on X_n. One has $\|P_n\| \to_{n\to\infty} 1$. In fact we will prove:

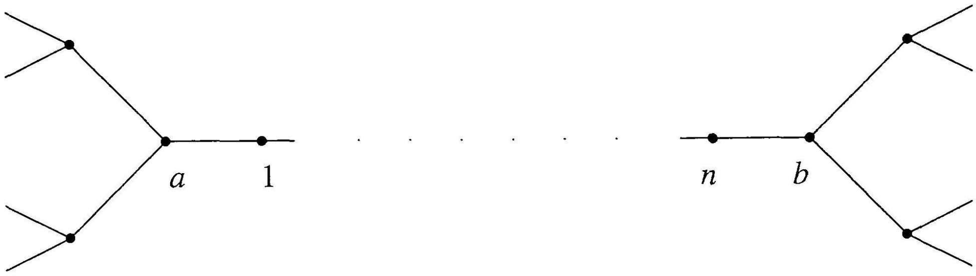

FIGURE 1
The graph X_n

PROPOSITION 2. *For $n \geq 7$ one has*

$$\|P_n\| > \cos\left(\frac{\pi}{n+3}\right) > \frac{2\sqrt{2}}{3}.$$

For any $n_0 \geq 1$ such that $\|P_{n_0}\| > \frac{2}{3}\sqrt{2}$ the eigenfunctions of P_{n_0} corresponding to the eigenvalue $\|P_{n_0}\|$ are in $l^2(X_{n_0}, N)$.

Proof. For $n \geq 7$ let $t = \sin\left(\frac{\pi}{n+3}\right) / \sin\left(\frac{2\pi}{n+3}\right)$ so that $0 < t < 1$.

For $x \in X \setminus I_n$ let $|x|$ be the minimum of its distances from a and b. We define the function f_n on X_n as follows:

$$f_n(y) = \begin{cases} t^{|y|} & \text{for } y \in X \setminus I_n \\ \sin\left(\frac{\pi(y+1)}{n+3}\right) / \sin\left(\frac{\pi}{n+3}\right) & \text{for } y = 1, \dots, n \\ 1 & \text{for } y = a, b. \end{cases}$$

We verify that

$$P_n f_n(i) = \cos\left(\frac{\pi}{n+3}\right) f_n(i) \qquad \text{for } i = 1, \dots, n$$

$$P_n f_n(x) = \frac{1}{3}\left(\cos^{-1}\left(\frac{\pi}{n+3}\right) + 2\cos\left(\frac{\pi}{n+3}\right)\right) f_n(x) \text{ for } x \in X_n \setminus \{1, \dots, n\}.$$

On the other hand for $n \geq 7$ we have $t < \frac{1}{\sqrt{3}}$ and

$$\sum_{x \in X_n \setminus I_n} f_n^2(x) N(x) = 2 \sum_{k=1}^{\infty} 2 \cdot 3^{k-1} \left(t^k\right)^2 \cdot 3 < \infty.$$

Thus f_n is in $l^2(X_n, N)$ and

$$P_n f_n \geq \cos\left(\frac{\pi}{n+3}\right) f_n.$$

So we have proved the first part of Proposition 2.

Let n_0 be such that

$$\|P_{n_0}\|_{l^2(X_{n_0},N)\to l^2(X_{n_0},N)} = \sigma > \frac{2\sqrt{2}}{3}\,.$$

Now let f be an eigenfunction of the operator P_{n_0} with the eigenvalue σ, i.e.

$$P_{n_0}f = \sigma f\,.$$

We want to show that $f \in l^2(X_{n_0}, N)$. Suppose this is not true, i.e.

$$\sum_{x\in X_{n_0}} f^2(x)N(x) = +\infty\,.$$

By Theorem 2, there exists a sequence of subsets of X_{n_0}, $A_k \subset X_{n_0}$ such that

$$\frac{\sum_{x\in\partial A_k} f^2(x)N(x)}{\sum_{x\in A_k} f^2(x)N(x)} \to_{k\to\infty} 0\,. \tag{3}$$

As I_{n_0} is a fixed finite set, the sequence $C_k = A_k \setminus I_{n_0}$ is non-empty for k sufficiently large. We need the following:

LEMMA 3. *One has*

$$\frac{\sum_{x\in\partial C_k} f^2(x)N(x)}{\sum_{x\in C_k} f^2(x)N(x)} \to_{k\to\infty} 0\,.$$

Proof. If $\sum_{x\in A_k} f^2(x)N(x) \to_{k\to\infty} \infty$ then the statement of the lemma is clear. Suppose then that for all k

$$\sum_{x\in A_k} f^2(x)N(x) \le \alpha < \infty\,. \tag{4}$$

If $A_k \cap I_{n_0} = \varnothing$ then A_k and C_k coincide. So we are interested only in those k for which $A_k \cap I_{n_0} \neq \varnothing$. Let us consider the ball B_R of radius R centered in $a \in I_{n_0}$ (*i.e.* those vertices in X_{n_0} for which at most R edges are needed to connect them to a).

Because of (3) and (4) we have that for k sufficiently large $\partial A_k \cap B_R = \varnothing$ which, by the fact that $A_k \cap I_{n_0} \neq \varnothing$, implies that $B_R \subset A_k$. But R can be chosen arbitrarily large and as f is not in $l^2(X,N)$ we get

$$\sum_{x\in A_k} f^2(x)N(x) \to_{k\to\infty} \infty\,,$$

which contradicts (4). This completes the proof of the lemma. □

On the subsets C_k the graphs X and X_{n_0} coincide. This implies:

$$\|P\|_{l^2(X,N)\to l^2(X,N)} \ge \sigma > \frac{2\sqrt{2}}{3}\,,$$

which yields the desired contradiction. This ends the proof of Proposition 2. □

4. NORMS OF RANDOM WALK OPERATORS

Now we will show how Theorem 3 can be used in the problem of computing the norm of the random walk operator P on some groups. Our strategy is as follows: we want to find a positive eigenfunction for the operator P which satisfies the generalized Følner condition. By Theorem 3 such an eigenfunction always exists and the eigenvalue corresponding to this eigenfunction is equal to the norm of the operator P. Theorem 3 is a particular case of Theorem 2 which can also be helpful in computing the norms of more general operators as shown in Section 4.3.

4.1 FREE GROUPS

First of all, as a simple illustration of this method, we will compute the norm of the simple random walk operator on free groups, which was first done by Kesten (see [9]) using a different method.

THEOREM 6 (Kesten). *Let Γ be the free group generated by the standard symmetric set of generators S. The norm of the simple random walk operator P associated to (Γ, S) is equal to*

$$\|P\| = \frac{2\sqrt{\#S - 1}}{\#S}\,.$$

Proof. The Cayley graph of (Γ, S) is a homogeneous tree T_k of degree $k = \#S$. We draw the tree T_k with level lines as in Figure 3 (level lines are marked by dotted lines). Let us choose arbitrarily a line as the line of level 0. We construct a function on vertices of this tree which depends only on the level of the vertex. For a vertex $v \in T_k$ we denote by $|v|$ its level. We define $f\colon T_k \to \mathbf{R}_+$ as follows

$$f(v) = \left(\frac{1}{\sqrt{k-1}}\right)^{|v|}.$$

One has

$$Pf = \frac{2\sqrt{k-1}}{k} f\,.$$

Let A_n be the set of vertices in T_k consisting of a chosen vertex e from the level 0 and the vertices lying below e up to the level n (in Figure 3 the vertices of A_2 are marked with circles). Then

$$\sum_{v \in A_n} f^2(v) = n+1\,,$$

$$\sum_{v \in \partial A_n} f^2(v) = 2\,.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence and by Theorem 3

$$\|P\| = \frac{2\sqrt{k-1}}{k}\,. \qquad \square$$

4.1.1 Remarks on generalized growth

Let Γ be a group generated by a finite, symmetric set S. For $id \neq \gamma \in \Gamma$ we define its length $|\gamma|$ as the minimal number of generators from S needed to represent γ, i.e.

$$|\gamma| = \min\{n\,;\ \gamma = s_{i_1} \dots s_{i_n}\,,\ s_{i_j} \in S\}\,,$$

and we declare $|id| = 0$.

The growth function (see [10], [18]) of the pair (Γ, S) associates to each integer $n \geq 0$ the number $\beta(\Gamma, S)(n)$ of elements $\gamma \in \Gamma$ such that $|\gamma| \leq n$, i.e.

$$\beta(\Gamma, S)(n) = \#\{\gamma \in \Gamma;\ |\gamma| \leq n\}\,.$$

One is often interested only in the type of the growth function. For instance, we say that the group Γ is of polynomial growth if there exist constants c and D such that

$$c^{-1} n^D \leq \beta(\Gamma, S)(n) \leq c n^D\,.$$

The exponent D does not depend on the set of generators S. If the growth function is bounded by a polynomial, it is known (see [6]) that Γ is of polynomial growth and D is an integer. For a group of polynomial growth with the exponent D, it is known (see [19]) that there exists a constant c such that

$$c^{-1} n^{-\frac{D}{2}} \leq P^{2n}(id, id) \leq c n^{-\frac{D}{2}}\,, \tag{5}$$

where $P^{2n}(id, id)$ is the probability of the return to the identity element of the simple random walk after $2n$ steps.

It seems natural to define a generalized growth function, using an eigenfunction of P. Let f be a positive eigenfunction of P corresponding to the eigenvalue equal to the norm of P, i.e.

$$Pf = \|P\| f\,.$$

The generalized growth function $\beta(\Gamma, S, f)$ associates to each positive integer n the number

$$\beta(\Gamma, S, f)(n) = \sum_{\gamma \in \Gamma, |\gamma| \leq n} f^2(\gamma),$$

i.e., each element in the ball of radius n is counted with weight f^2.

Let us compute the generalized growth function in a particular case. Let P be the simple random walk operator on the free group with the standard set of generators of cardinality k as in Section 4.1. Let g be the unique radial eigenfunction of P corresponding to the eigenvalue $\|P\|$ and such that $g(id) = 1$. Explicitly we have:

$$g(\gamma) = \left(\frac{k-2}{k}|\gamma| + 1\right)\left(\frac{1}{\sqrt{k-1}}\right)^{|\gamma|}.$$

Then we have

$$\sum_{\gamma \in \Gamma, |\gamma| \leq n} g^2(\gamma) =$$

$$n^3\left(\frac{k^2-4k+4}{3k^2-3k}\right) + n^2\left(\frac{3k^2-8k+4}{2k^2-2k}\right) + n\left(\frac{7k^2-16k+4}{6k^2-6k}\right) + 1.$$

This shows that the generalized growth is like n^3. In particular the sequence of balls is a generalized Følner sequence.

By analogy to (5) we conjecture that the fact that the generalized growth function for the free groups is like n^3 explains that for the free groups one has (see [16]):

$$c^{-1}\lambda^{2n} n^{-\frac{3}{2}} \leq P^{2n}(id, id) \leq c\lambda^{2n} n^{-\frac{3}{2}},$$

where c is a constant and λ is the norm of P.

4.2 Free products of finite groups

Random walks on free products of finite groups were already considered in [1], [3], [17] and [21].

Let us consider the group $\mathbf{Z}_m \star \mathbf{Z}_n$ with the following generating set:

- if $m \neq 2$ we take $\{\pm 1\}$ as generators of $\mathbf{Z}_m = \{0, 1, \ldots, m-1\}$;
- we take $\{1\}$ as a generator of $\mathbf{Z}_2 = \{0, 1\}$.

In Figure 2 we represent the Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the above set of generators. In general the Cayley graph for $\mathbf{Z}_m \star \mathbf{Z}_n$ with the generating set defined above has the following construction:

- m-gons and n-gons are attached to each other;
- at each vertex of an n-gon there is one m-gon attached and at each vertex of an m-gon there is one n-gon attached.

4.2.1 $\mathbf{Z}_2 \star \mathbf{Z}_4$

We will present our method in the special case for $\mathbf{Z}_2 \star \mathbf{Z}_4$. The Cayley graph for this group is represented in Figure 2. Our aim is to construct the eigenfunction f of the random walk operator satisfying the generalized Følner condition. By Theorem 3, the eigenvalue corresponding to this eigenfunction is equal to the norm of a random walk operator. We will construct f in two steps.

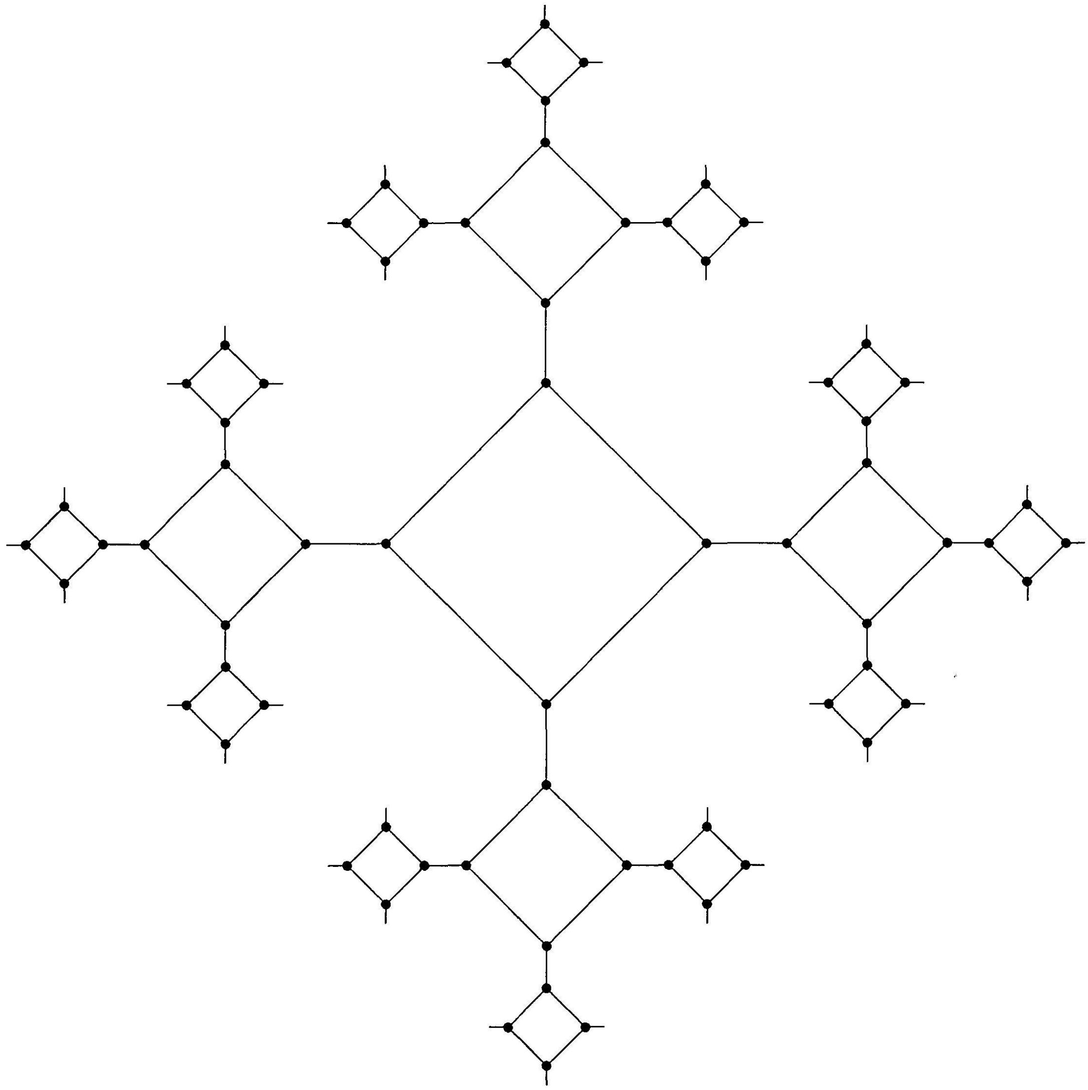

FIGURE 2
Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$

STEP 1. If we contract the squares to points, the Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$ is deformed to the homogeneous tree T_4 of order 4 (each vertex has 4 neighbors), which is represented in Figure 3. First of all we construct a function on vertices of T_4 satisfying the generalized Følner condition.

We draw the graph T_4 as in Figure 3, i.e. with one point set apart at infinity. The level lines or horocycles are marked by dotted lines. Each vertex of T_4 has one neighbor above and three neighbors below.

Let us fix two positive numbers r, s and define the positive function g on the vertices of the tree T_4

$$g\colon (\text{vertices of } T_4) \to \mathbf{R}_+$$

as follows:

if w is a neighbor of v lying below v then (see Figure 4)

(1) $g(w) = rg(v)$ if w is the right or left neighbor;

(2) $g(w) = sg(v)$ if w is the middle neighbor.

The above defines the function g up to a constant. Let us fix one vertex e (for instance lying on the horocycle of level 0) and put $g(e) = 1$.

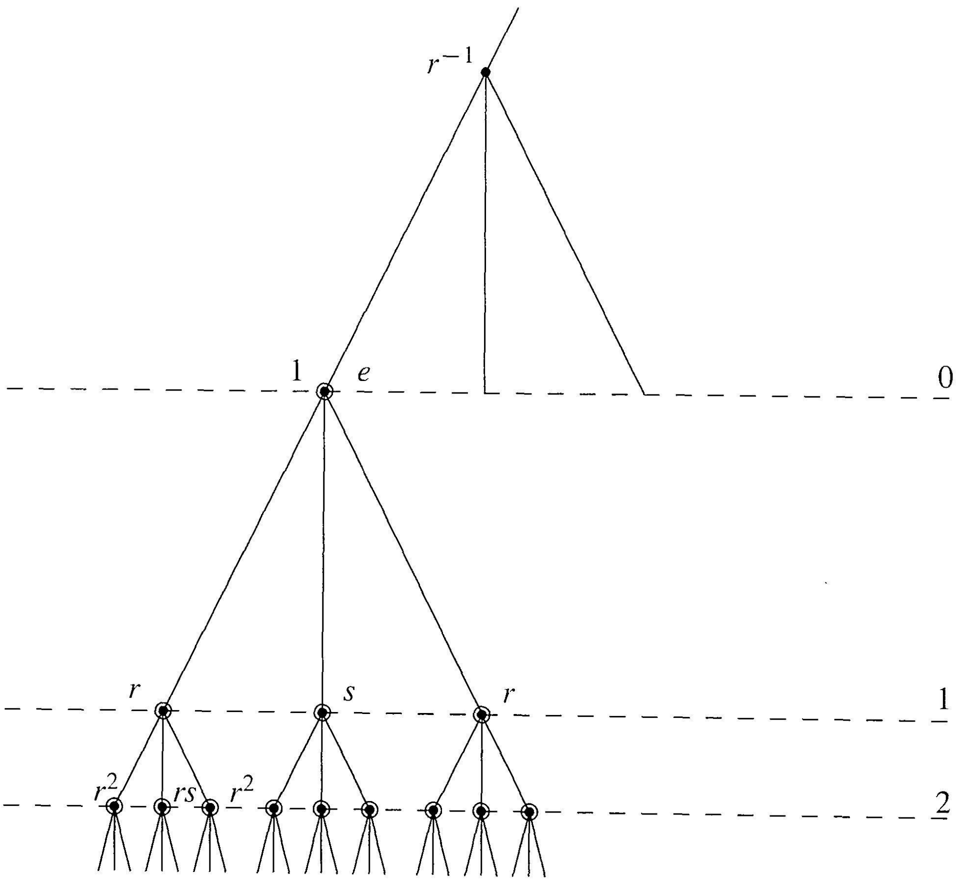

FIGURE 3
Tree T_4 of order 4

Now we need

LEMMA 4. *For* $2r^2 + s^2 = 1$ *the function* g *satisfies the generalized Følner condition, i.e. there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* T_4 *such that*

$$\frac{\sum_{v \in \partial A_n} g^2(v)}{\sum_{v \in A_n} g^2(v)} \to_{n \to \infty} 0\,.$$

Proof. Let A_n be the subset of vertices of the tree T_4 consisting of e and the vertices lying below e up to the level n (in Figure 3 the vertices of A_2 are marked with circles).

One can easily see that

$$\sum_{v \in A_n} g^2(v) = n + 1\,,$$

$$\sum_{v \in \partial A_n} g^2(v) = 2\,.$$

Thus $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence for P corresponding to g. □

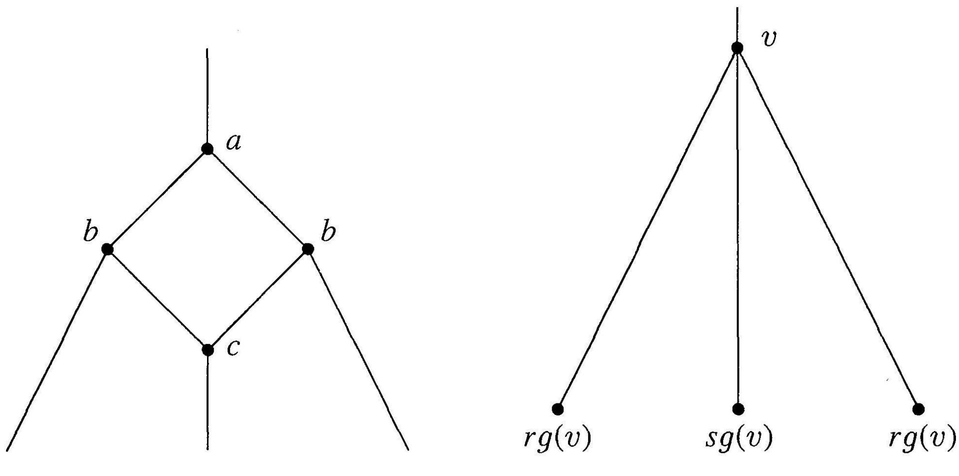

FIGURE 4

Labelling of vertices and the definition of the function g

STEP 2. The second step consists of labelling the vertices of the Cayley graph of $\mathbf{Z}_2 \star \mathbf{Z}_4$ with a, b or c (the precise values of the numbers a, b and c are given later). The vertices of each square are labelled as in Figure 4. This defines the unique labelling if we bear in mind the way we have drawn the tree T_4 obtained by contracting the squares (see Figure 3).

Now we can define the positive function f on $\mathbf{Z}_2 \star \mathbf{Z}_4$ as follows. If v is the vertex of type t ($t = a$, b or c) of the square which corresponds to the vertex w of the tree T_4 then

$$f(v) = tg(w).$$

We want to find a, b, c, r, s and λ so that f is an eigenfunction of the random walk operator P with the eigenvalue λ.

Let us write the equation

$$Pf = \lambda f$$

for vertices of type a, b and c. On a vertex of type a, the function f has to satisfy the following

(6) $$\frac{b + 2br}{3} = \lambda ar,$$

(7) $$\frac{c + 2bs}{3} = \lambda as.$$

For a vertex of type b, function f has to satisfy

(8) $$\frac{a + c + ar}{3} = \lambda b$$

and for a vertex of type c, function f has to satisfy

(9) $$\frac{2b + as}{3} = \lambda c.$$

If f satisfies the above conditions it is an eigenfunction of P with the eigenvalue λ. For $2r^2 + s^2 = 1$, by Lemma 4 the function g satisfies the generalized Følner condition and so does f. So we want to have a condition

(10) $$2r^2 + s^2 = 1.$$

After solving equations (6)-(10) we obtain the following values for a, b, c, r, s and λ (a, b and c are determined up to a constant so we suppose a=1):

$$a = 1; \qquad b = \frac{u\sqrt{1 - 2u^2}}{-1 + 4u^2}; \qquad c = \frac{1 - 2u^2}{-1 + 4u^2};$$

$$r = u; \qquad s = \sqrt{1 - 2u^2}; \qquad \lambda = \frac{-1 + 2u + 4u^2}{3\sqrt{1 - 2u^2}};$$

where

$$u = \frac{\sqrt{33} - 1}{8}.$$

For the above values, f is an eigenfunction of the operator P and satisfies the generalized Følner condition. By Theorem 3 the norm of the random walk operator on $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the generating subset as defined before is then equal to

$$\|P\| = \frac{\sqrt{33}+7}{\sqrt{\sqrt{33}-1}} \approx 0.98 .$$

4.2.2 General case

The idea presented for $\mathbf{Z}_2 \star \mathbf{Z}_4$ can be used in the general case for $\mathbf{Z}_n \star \mathbf{Z}_m$. As the solution involves roots of some polynomial of degree nm, we will not give details.

4.3 Mean operator on the hyperbolic plane

Let us consider the hyperbolic upper half-plane $H = \{z = x + iy \in \mathbf{C}; x \in \mathbf{R}, y > 0\}$ with a Riemannian metric $d_H z = \frac{\sqrt{dx^2+dy^2}}{y}$ which gives rise to the measure $\mu_H = \frac{dxdy}{y^2}$. We consider the operator P,

$$Pf(z_0) = \int_{|z-z_0|=R} f(z)\, dm_R(z) ,$$

where dm_R is a uniform probability measure on a hyperbolic circle of radius R. We want to compute the norm of the operator P acting on $L^2(H, d_H z)$.

First of all let us remark that the function:

$$f(z) = \sqrt{\mathrm{Im(z)}}, \tag{11}$$

is an eigenfunction of P. An easy way to see this is to note that P commutes with isometries of H and that the isometries consisting of horizontal translations and homotheties act transitively on H. The effect of these on the function f is that they just multiply it by a constant.

Now we would like to show that one can find a Følner sequence with respect to the function f. Let us consider a sequence $\{A_n\}_{n=1}^{\infty}$ of rectangles (in the Euclidean sense) in H:

$$A_n = \{z \in H ; e^{-n} \le \mathrm{Im}(z) \le 1,\ 0 \le \mathrm{Re}(z) \le n\} .$$

It is easy to see that the measure $|\partial A_n|$ of the boundary of A_n is bounded by the measure of the following set B_n (see Figure 5):

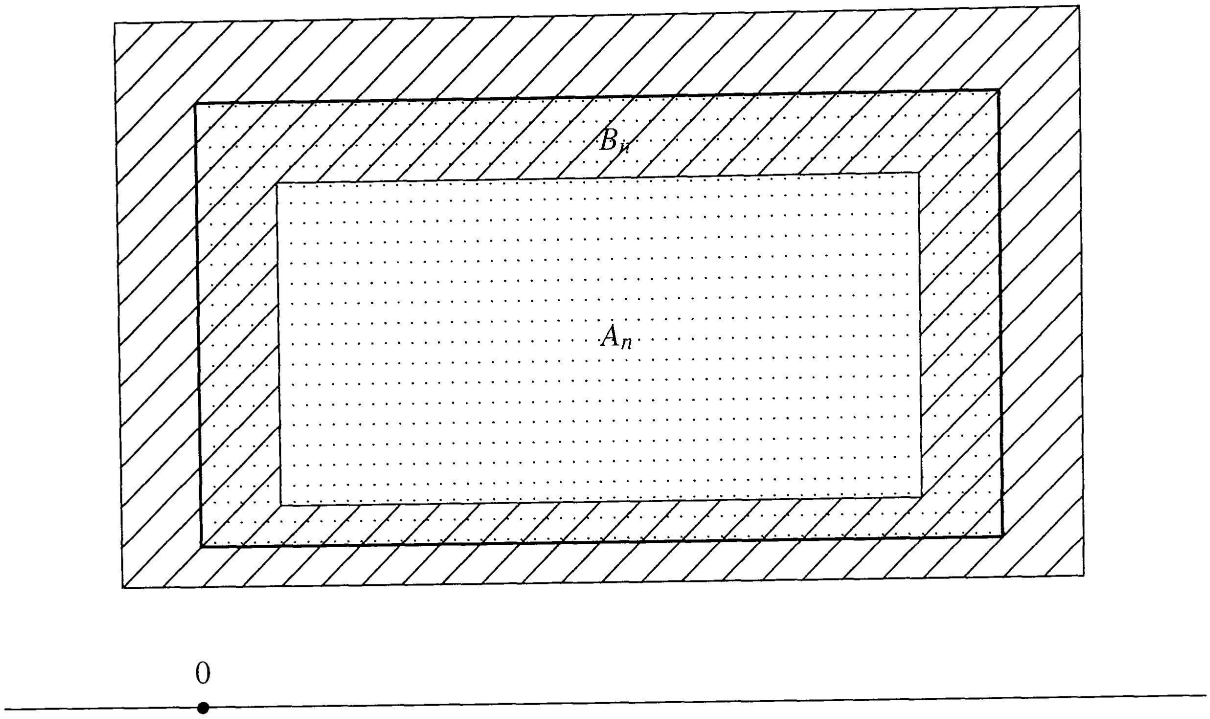

FIGURE 5
Sets A_n and B_n

$$\begin{aligned} B_n = &\{z \in H;\ -R \leq \mathrm{Re}(z) \leq R,\ e^R \geq \mathrm{Im}(z) \geq e^{-n-R}\} \\ &\cup \{z \in H;\ -R+n \leq \mathrm{Re}(z) \leq n+R,\ e^R \geq \mathrm{Im}(z) \geq e^{-n-R}\} \\ &\cup \{z \in H;\ -R \leq \mathrm{Re}(z) \leq n+R,\ e^R \geq \mathrm{Im}(z) \geq e^{-R}\} \\ &\cup \{z \in H;\ -R \leq \mathrm{Re}(z) \leq n+R,\ e^{-n+R} \geq \mathrm{Im}(z) \geq e^{-n-R}\}. \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence. Thus

$$\|P\|_{L^2(H,d_H z) \to L^2(H,d_H z)} = \int_{|z-i|=R} \sqrt{\mathrm{Im(z)}}\, dm_R(z).$$

4.4 WREATH PRODUCTS

Let G and F be finitely generated groups. We define the wreath product $G \wr F$ of these groups as follows. Elements of $G \wr F$ are couples (g, γ_1) where $g \colon F \to G$ is a function such that $g(\gamma)$ is different from the identity element id_G of G only for finitely many elements γ in F, and where γ_1 is an element of F. The multiplication in $G \wr F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(\gamma) = g_1(\gamma)g_2(\gamma\gamma_1) \quad \text{for } \gamma \in F\,.$$

If S_G and S_F are generators of G and F respectively then

$$\{(g,\gamma)\ ;\ (g(F) = id_G,\ \gamma \in S_F) \text{ or } (g(F \setminus id_F) = id_G,\ g(id_F) \in S_G,\ \gamma = id_F)\}$$

is a generating subset for $G \wr F$.

Let μ and ν be symmetric, finitely supported probability measures on F and G respectively.

As there is a natural embedding of F and G into $G \wr F$, one can view the measures μ and ν as measures on $G \wr F$. More precisely:

$$\nu(g,\gamma) = \begin{cases} \nu(g(id_F)) & \text{if } \gamma = id_F \text{ and } g(F \setminus id_F) = id_G \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu(g,\gamma) = \begin{cases} \mu(\gamma) & \text{if } g(F) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \star \nu \star \mu$ is a symmetric measure on $G \wr F$. Explicitly we have:

$$\mu \star \nu \star \mu(g,\gamma) = \begin{cases} \mu(\gamma(\gamma_0)^{-1})\mu(\gamma_0)\nu(g(\gamma_0)) & \text{if } g(F \setminus \gamma_0) = id_G \\ 0 & \text{otherwise}\,. \end{cases}$$

We want to prove:

THEOREM 7. *Let F and G be finitely generated groups. If F is amenable then the spectral radius of ν on G is the same as the spectral radius of $\mu \star \nu \star \mu$ on $G \wr F$.*

Proof. We will prove Theorem 7 by constructing on $G \wr F$ a positive function $\widetilde{f}$ which is an eigenfunction for the convolution by $\mu \star \nu \star \mu$ with eigenvalue $\|\nu\|_{l^2(G)\to l^2(G)}$ and for which there exists a generalized Følner sequence.

Let f be a positive eigenfunction for the operator which is a convolution on $l^2(G)$ by ν, corresponding to the eigenvalue $\|\nu\|$, i.e.

$$f \star \nu = \|\nu\| f\,. \tag{12}$$

We can normalize f so that

$$f(id_G) = 1\,. \tag{13}$$

By Theorem 3 (and the remark after its proof) there exists a sequence of finite subsets $A_n \subset G$, such that

$$\frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} \longrightarrow_{n\to\infty} 0\,.$$

As the group F is amenable there exists a sequence of finite subsets $B_n \subset F$, such that

$$\frac{\#\partial B_n}{\#B_n} \longrightarrow_{n\to\infty} 0\,.$$

For technical reasons let us choose the sequences B_n and A_n in such a way that

$$\frac{\#\partial B_n}{\#B_n} < \frac{1}{n} \quad \text{and} \quad \frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} < \frac{1}{n(\#B_n)}\,. \tag{14}$$

Now, on $G \wr F$ we define $\widetilde{f}$ as follows

$$\widetilde{f}(g,\gamma_1) = \prod_{\gamma\in F} f(g(\gamma))\,.$$

The function $\widetilde{f}$ is well defined because by (13), $f(g(\gamma))$ is different from 1 only for finitely many $\gamma \in F$. This function is of course positive and does not depend on γ_1. From (12) one has

$$\widetilde{f} \star \mu \star \nu \star \mu = \widetilde{f} \star \nu \star \mu = \|\nu\| \widetilde{f} \star \mu = \|\nu\| \widetilde{f}\,.$$

To complete the proof of Theorem 7 it is enough to construct a generalized Følner sequence $C_n \subset G \wr F$ for $\widetilde{f}$. We define C_n as follows:

$$C_n = \{(g,\gamma_1);\gamma_1 \in B_n,\ g(B_n) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\}\,.$$

LEMMA 5. *The sequence $C_n \subset G \wr F$ is a generalized Følner sequence for $\widetilde{f}$.*

Proof. Let us define sets D_n and ∂D_n as follows:

$$\begin{aligned} D_n &= \{g\colon F \to G;\ g(B_n) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\}\,,\\ \partial D_n &= \{g\colon F \to G;\ \text{there exists } \gamma_0 \in B_n \text{ such that } g(\gamma_0) \in \partial A_n,\\ &\qquad g(B_n \setminus \gamma_0) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\}\,. \end{aligned}$$

Thus

$$\begin{aligned} C_n &= D_n \times B_n,\\ \partial C_n &= (\partial D_n \times B_n) \cup (D_n \times \partial B_n)\,. \end{aligned}$$

We have then

$$\sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2 = \sum_{(g,\gamma_1)\in C_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$

$$= \sum_{(g,\gamma_1)\in D_n\times B_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2 = \#B_n \sum_{g\in D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2.$$

On the other hand

$$\sum_{(g,\gamma_1)\in \partial C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2 = \sum_{(g,\gamma_1)\in \partial C_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$

$$= \sum_{(g,\gamma_1)\subset(\partial D_n\times B_n)\cup(D_n\times\partial B_n)} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$

$$= \#\partial B_n \sum_{g\in D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2 + \#B_n \sum_{g\in \partial D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$

$$= \frac{\#\partial B_n}{\#B_n} \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2$$

$$+ \frac{\sum_{g\in\partial D_n}\left(\prod_{\gamma\in F} f(g(\gamma))\right)^2}{\sum_{g\in D_n}\left(\prod_{\gamma\in F} f(g(\gamma))\right)^2} \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2.$$

But

$$\sum_{g\in\partial D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2 = \#B_n \frac{\sum_{\alpha\in\partial A_n} f^2(\alpha)}{\sum_{\alpha\in A_n} f^2(\alpha)} \sum_{g\in D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2.$$

Thus by (14)

$$\sum_{(g,\gamma_1)\in \partial C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2 = \left(\frac{\#\partial B_n}{\#B_n} + \#B_n \frac{\sum_{\alpha\in\partial A_n} f^2(\alpha)}{\sum_{\alpha\in A_n} f^2(\alpha)}\right) \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2$$

$$\leq \frac{2}{n} \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2,$$

which shows that C_n is a generalized Følner sequence for $\widetilde{f}$. □

This ends the proof of Theorem 7.

5. Lower bounds

Now we will consider generalized Følner sequences for functions f such that

$$Pf \geq \|P\| f\,.$$

This will enable us to obtain some lower bounds on the norm of random walk operators on graphs.

As in Section 3, let X be a connected, locally finite graph and let P be the simple random walk operator on X.

In this section we will prove the following lower bound on the norm $\|P\|$:

THEOREM 8. *Let X be a graph such that at each vertex there are at most k edges. Then*

$$\|P\| \geq \frac{2\sqrt{k-1}}{k}\,.$$

The norm of the random walk operator $\|P\|$ is equal to $\frac{2\sqrt{k-1}}{k}$ for the random walk on the tree which has k edges at each vertex. In [9] Kesten proved this lower bound in the case of Cayley graphs.

Proof of Theorem 8. Let us consider a graph X such that at each vertex there are at most k edges. We can suppose that $k \geq 3$ because for $k = 2$ we obtain subgraphs of $\mathbf{Z}$ or finite graphs, and necessarily $\|P\| = 1$. As it is enough to prove the desired bound for any connected component of X, we can suppose that X is connected.

In order to show that $\|P\|$ is large enough, we will construct a sequence of functions $f_n \in l^2(X, N)$ such that

$$\limsup_{n\to+\infty} \frac{\|Pf_n\|_{l^2(X,N)}}{\|f_n\|_{l^2(X,N)}} \geq \frac{2\sqrt{k-1}}{k}\,.$$

Let us endow the set of vertices of X with a metric. The distance between two vertices is the smallest number of edges needed to connect them. Let us choose a vertex e in X and for a vertex v let us denote by $|v|$ its distance from e.

Let f be the unique (up to translations and multiplications) radial eigenfunction of P on the homogeneous tree of degree k, corresponding to the eigenvalue $\frac{2\sqrt{k-1}}{k}$, which is the norm of P on this tree, i.e.

$$f(v) = g(|v|) = \left(\frac{k-2}{k}|v|+1\right)\left(\frac{1}{\sqrt{k-1}}\right)^{|v|}. \tag{15}$$

Using (15) we can define f on X. We then prove

LEMMA 6. *For any vertex* $v \in X$,

$$Pf(v) \geq \frac{2\sqrt{k-1}}{k} f(v).$$

Proof. If $v = e$ the result is clearly true. Let us consider then a vertex $v \in X$ such that $n = |v| \geq 1$. Let the number of neighbors of v which are at a distance $n-1$ or n from e be equal respectively to p and q. So the number of neighbors of v which are at a distance $n+1$ is equal to $N(v)-p-q$. Hence

$$Pf(v) = \frac{1}{N(v)}\left(pg(n-1)+qg(n)+(N(v)-p-q)g(n+1)\right).$$

As $p \geq 1$ and g is a decreasing function,

$$Pf(v) \geq \frac{1}{N(v)}\left(g(n-1)+(N(v)-1)g(n+1)\right).$$

As $N(v) \leq k$ and $g(n-1) \geq g(n+1)$,

$$Pf(v) \geq \frac{1}{k}\left(g(n-1)+(k-1)g(n+1)\right) = \frac{2\sqrt{k-1}}{k} g(n). \qquad \square$$

Let us denote by S_n and B_n the vertices which are respectively at a distance n and less than or equal to n.

LEMMA 7.

$$\frac{\sum_{v\in S_{n+1}} f^2(v)N(v)}{\sum_{v\in B_n} f^2(v)N(v)} \to_{n\to\infty} 0.$$

Proof. As $1 \leq N(v) \leq k$ it is enough to show that

$$\frac{\sum_{v\in S_{n+1}} f^2(v)}{\sum_{v\in B_n} f^2(v)} \to_{n\to\infty} 0.$$

Let us denote

$$a_n = \sum_{v\in S_n} f^2(v) = |S_n| g^2(n).$$

As $|S_{n+1}| \leq (k-1)|S_n|$ one has

$$(16)\ a_{n+1} = |S_{n+1}| g^2(n+1) \leq (k-1)|S_n| g^2(n+1) = \left(1 + \frac{k-2}{(k-2)n+k}\right)^2 a_n\,.$$

We have to show that

$$(17) \qquad \frac{\sum_{v \in S_{n+1}} f^2(v)}{\sum_{v \in B_n} f^2(v)} = \frac{a_{n+1}}{a_1 + \cdots + a_n} \to_{n\to\infty} 0\,.$$

It is a standard exercise to show that (16) implies (17). □

Let f_n be the sequence of functions which are restrictions of f to the vertices that are at a distance not greater than n :

$$f_n = f|_{B_n}\,.$$

By Lemma 6 and Lemma 7 it follows that

$$\limsup_{n\to+\infty} \frac{\|Pf_n\|_{l^2(X,N)}}{\|f_n\|_{l^2(X,N)}} \geq \frac{2\sqrt{k-1}}{k}\,,$$

which proves Theorem 8. □

Some examples of upper bounds on the norm of the simple random walk operator on graphs and their comparison with the lower bound from Theorem 8 can be found in [22].

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