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Artikel: GENERALIZED FØLNER CONDITION AND THE NORMS OF RANDOM WALK OPERATORS ON GROUPS

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Kapitel: 3.1 Existence of eigenfunctions

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This implies that

$$\frac{1}{\#S}|A|_{f^2} = \sum_{\gamma \in A} f^2(\gamma),$$

$$\frac{1}{\lambda(\#S)^2}|\partial A|_{f^2} \leq \sum_{\gamma \in \partial A} f^2(\gamma) \leq \lambda |\partial A|_{f^2}.$$

By Theorem 2, the first condition implies the second one. □

REMARK. The proof of Theorem 3 can easily be generalized to the case where P is a convolution operator with a finitely supported probability measure.

3. REMARKS

We will now make some comments about Theorems 2 and 3. We will state some theorems about the existence of eigenfunctions for the Markov operator and discuss whether one can take in the generalized Følner condition the eigenfunctions to be in $L^2(X,\mu)$.

For simplicity we will suppose that X is a connected, locally finite graph (i.e. the degree of each vertex is finite) and we consider the *simple random walk* going with equal probability from one vertex to any of its neighbors. We associate with this random walk the simple random walk operator P defined by

$$Pf(v) = \frac{1}{N(v)} \sum_{w \sim v} f(w) \quad \text{for } f \in l^2(X,N)$$

where $N(v)$ is the degree of vertex v in X (*i.e.* the number of edges adjacent to v), where $w \sim v$ means that w and v are connected by an edge and where $l^2(X,N)$ is the space of real-valued functions f on the vertices of X such that $\sum_{x \in X} f^2(x)N(x)$ is finite.

3.1 EXISTENCE OF EIGENFUNCTIONS

THEOREM 4 ([20]). *Let X be an infinite, locally finite graph and let P be the simple random walk operator on $l^2(X,N)$. For any $\lambda \geq \|P\|$ there exists a positive eigenfunction f of P with eigenvalue λ, i.e.*

$$Pf(x) = \lambda f(x) \quad \text{and } f(x) > 0 \text{ for } x \in X.$$

For $\lambda < \|P\|$ there are no positive eigenfunctions of P with eigenvalue λ.

Proof. There are several proofs of this theorem. In [20] one can find the proof where the analogue of Perron-Frobenius theory is developed and in [11] the truncation method is used. □

3.2 Eigenfunctions in l^2

One can ask whether the positive eigenfunctions of the random walk operator are in $l^2(X,N)$. The answer is no in the case when X is the Cayley graph of an infinite group Γ (see Theorem 5). But in the general case there are examples of eigenfunctions which are in $l^2(X,N)$ (see Proposition 2).

3.2.1 The case of groups

THEOREM 5. *Let f be a positive eigenfunction of the simple random walk operator P on the group Γ generated by a finite symmetric set S, i.e. $Pf = \lambda f$. If Γ is infinite then*

$$\sum_{\gamma\in\Gamma} f^2(\gamma) = +\infty .$$

Proof. Suppose the contrary, i.e. that there is a positive eigenfunction f of the operator P for which the l^2 norm is finite:

$$Pf_0 = \lambda f_0 ,$$
$$\sum_{\gamma\in\Gamma} f_0^2(\gamma) < +\infty .$$

The second condition implies that f_0 is not constant and so there are $\gamma_0, \gamma_1 \in \Gamma$ such that

$$f_0(\gamma_0) < f_0(\gamma_1) .$$

Let us define the function f_1 as a translation of f_0 by $\gamma_0\gamma_1^{-1}$, i.e.

$$f_1(\gamma) = f_0(\gamma_0\gamma_1^{-1}\gamma) .$$

The function f_1, being the translation of f_0, is an eigenfunction of P, i.e.

$$Pf_1 = \lambda f_1 .$$

So the function $\widetilde{f}$ defined as follows:

$$\widetilde{f}(\gamma) = \max\{f_0(\gamma), f_1(\gamma)\} ,$$

satisfies

$$P\widetilde{f} \geq \lambda \widetilde{f} .$$