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Proof. There are several proofs of this theorem. In [20] one can find the proof where the analogue of Perron-Frobenius theory is developed and in [11] the truncation method is used. □

3.2 Eigenfunctions in l^2

One can ask whether the positive eigenfunctions of the random walk operator are in $l^2(X,N)$. The answer is no in the case when X is the Cayley graph of an infinite group Γ (see Theorem 5). But in the general case there are examples of eigenfunctions which are in $l^2(X,N)$ (see Proposition 2).

3.2.1 The case of groups

THEOREM 5. *Let f be a positive eigenfunction of the simple random walk operator P on the group Γ generated by a finite symmetric set S, i.e. $Pf = \lambda f$. If Γ is infinite then*

$$\sum_{\gamma\in\Gamma} f^2(\gamma) = +\infty .$$

Proof. Suppose the contrary, i.e. that there is a positive eigenfunction f of the operator P for which the l^2 norm is finite:

$$Pf_0 = \lambda f_0 ,$$

$$\sum_{\gamma\in\Gamma} f_0^2(\gamma) < +\infty .$$

The second condition implies that f_0 is not constant and so there are $\gamma_0, \gamma_1 \in \Gamma$ such that

$$f_0(\gamma_0) < f_0(\gamma_1) .$$

Let us define the function f_1 as a translation of f_0 by $\gamma_0\gamma_1^{-1}$, i.e.

$$f_1(\gamma) = f_0(\gamma_0\gamma_1^{-1}\gamma) .$$

The function f_1, being the translation of f_0, is an eigenfunction of P, i.e.

$$Pf_1 = \lambda f_1 .$$

So the function $\widetilde{f}$ defined as follows:

$$\widetilde{f}(\gamma) = \max\{f_0(\gamma), f_1(\gamma)\} ,$$

satisfies

$$P\widetilde{f} \geq \lambda \widetilde{f} .$$

As f_0 and f_1 are in $l^2(\Gamma)$, the function $\widetilde{f}$ is in $l^2(\Gamma)$ as well. The functions f_0 and f_1 have the same l^2 norms and

$$f_1(\gamma_1) = f_0(\gamma_0) < f_0(\gamma_1),$$

so there exists $\gamma_2 \in \Gamma$ such that

$$f_1(\gamma_2) > f_0(\gamma_2).$$

Note that these two inequalities imply that $\widetilde{f} \geq f_0$ with equality at some points and strict inequality at some other points. Thus $g = \widetilde{f} - f_0$ satisfies $g \geq 0$, $g \neq 0$, g vanishes at some points and $Pg \geq \lambda g$. Let us prove that this implies $P\widetilde{f} \neq \lambda\widetilde{f}$. Indeed, if we had equality then $Pg = \lambda g$ as well and thus $P^n g = \lambda^n g$. Taking n large enough makes $P^n g$ non-zero at points where g vanishes, a contradiction. We have thus shown that $P\widetilde{f} \geq \lambda\widetilde{f}$ with $P\widetilde{f} \neq \lambda\widetilde{f}$.

This means that

$$\|P\widetilde{f}\|_{l^2(\Gamma)} > \lambda\|\widetilde{f}\|_{l^2(\Gamma)}.$$

Hence

$$\|P\| > \lambda.$$

But this provides the desired contradiction because by Theorem 4 there are no positive eigenfunctions of P with an eigenvalue smaller than the norm of P. □

3.2.2 The general case

It will be shown that there are examples of the infinite graph X and the simple random walk operator P for which there is a positive eigenfunction in $l^2(X,N)$. It was pointed out to us by the referee that when P is the adjacency operator, examples of infinite graphs with positive eigenvalues in l^2 can be found for instance in [5] (page 232).

Let X be a uniform tree (*i.e.* a simply connected graph) of degree 3. By a theorem of Kesten (see [9]) one knows that $\|P\| = \frac{2}{3}\sqrt{2} < 1$. Let a and b be two neighboring vertices in X. Now let X_n be a graph which is the same as the graph X, except that the edge (a,b) is subdivided into n vertices. Let I_n denote the set of vertices a, b and added vertices which we label $1,\ldots,n$ (see Figure 1). Let P_n be the simple random walk operator on X_n. One has $\|P_n\| \to_{n\to\infty} 1$. In fact we will prove:

FIGURE 1
The graph X_n

PROPOSITION 2. *For $n \geq 7$ one has*

$$\|P_n\| > \cos\left(\frac{\pi}{n+3}\right) > \frac{2\sqrt{2}}{3}.$$

For any $n_0 \geq 1$ such that $\|P_{n_0}\| > \frac{2}{3}\sqrt{2}$ the eigenfunctions of P_{n_0} corresponding to the eigenvalue $\|P_{n_0}\|$ are in $l^2(X_{n_0}, N)$.

Proof. For $n \geq 7$ let $t = \sin\left(\frac{\pi}{n+3}\right)/\sin\left(\frac{2\pi}{n+3}\right)$ so that $0 < t < 1$.

For $x \in X \setminus I_n$ let $|x|$ be the minimum of its distances from a and b. We define the function f_n on X_n as follows:

$$f_n(y) = \begin{cases} t^{|y|} & \text{for } y \in X \setminus I_n \\ \sin\left(\frac{\pi(y+1)}{n+3}\right)/\sin\left(\frac{\pi}{n+3}\right) & \text{for } y = 1, \dots, n \\ 1 & \text{for } y = a, b. \end{cases}$$

We verify that

$$P_n f_n(i) = \cos\left(\frac{\pi}{n+3}\right) f_n(i) \qquad \text{for } i = 1, \dots, n$$

$$P_n f_n(x) = \frac{1}{3}\left(\cos^{-1}\left(\frac{\pi}{n+3}\right) + 2\cos\left(\frac{\pi}{n+3}\right)\right) f_n(x) \text{ for } x \in X_n \setminus \{1, \dots, n\}.$$

On the other hand for $n \geq 7$ we have $t < \frac{1}{\sqrt{3}}$ and

$$\sum_{x \in X_n \setminus I_n} f_n^2(x) N(x) = 2 \sum_{k=1}^{\infty} 2 \cdot 3^{k-1} \left(t^k\right)^2 \cdot 3 < \infty.$$

Thus f_n is in $l^2(X_n, N)$ and

$$P_n f_n \geq \cos\left(\frac{\pi}{n+3}\right) f_n.$$

So we have proved the first part of Proposition 2.

Let n_0 be such that

$$\|P_{n_0}\|_{l^2(X_{n_0},N)\to l^2(X_{n_0},N)} = \sigma > \frac{2\sqrt{2}}{3}\,.$$

Now let f be an eigenfunction of the operator P_{n_0} with the eigenvalue σ, i.e.

$$P_{n_0}f = \sigma f\,.$$

We want to show that $f \in l^2(X_{n_0}, N)$. Suppose this is not true, i.e.

$$\sum_{x\in X_{n_0}} f^2(x)N(x) = +\infty\,.$$

By Theorem 2, there exists a sequence of subsets of X_{n_0}, $A_k \subset X_{n_0}$ such that

$$\frac{\sum_{x\in\partial A_k} f^2(x)N(x)}{\sum_{x\in A_k} f^2(x)N(x)} \to_{k\to\infty} 0\,. \tag{3}$$

As I_{n_0} is a fixed finite set, the sequence $C_k = A_k \setminus I_{n_0}$ is non-empty for k sufficiently large. We need the following:

LEMMA 3. *One has*

$$\frac{\sum_{x\in\partial C_k} f^2(x)N(x)}{\sum_{x\in C_k} f^2(x)N(x)} \to_{k\to\infty} 0\,.$$

Proof. If $\sum_{x\in A_k} f^2(x)N(x) \to_{k\to\infty} \infty$ then the statement of the lemma is clear. Suppose then that for all k

$$\sum_{x\in A_k} f^2(x)N(x) \leq \alpha < \infty\,. \tag{4}$$

If $A_k \cap I_{n_0} = \varnothing$ then A_k and C_k coincide. So we are interested only in those k for which $A_k \cap I_{n_0} \neq \varnothing$. Let us consider the ball B_R of radius R centered in $a \in I_{n_0}$ (*i.e.* those vertices in X_{n_0} for which at most R edges are needed to connect them to a).

Because of (3) and (4) we have that for k sufficiently large $\partial A_k \cap B_R = \varnothing$ which, by the fact that $A_k \cap I_{n_0} \neq \varnothing$, implies that $B_R \subset A_k$. But R can be chosen arbitrarily large and as f is not in $l^2(X, N)$ we get

$$\sum_{x\in A_k} f^2(x)N(x) \to_{k\to\infty} \infty\,,$$

which contradicts (4). This completes the proof of the lemma. □

On the subsets C_k the graphs X and X_{n_0} coincide. This implies:

$$\|P\|_{l^2(X,N)\to l^2(X,N)} \geq \sigma > \frac{2\sqrt{2}}{3}\,,$$

which yields the desired contradiction. This ends the proof of Proposition 2. □