4.2 Free products of finite groups

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The generalized growth function $\beta(\Gamma, S, f)$ associates to each positive integer n the number

$$\beta(\Gamma, S, f)(n) = \sum_{\gamma \in \Gamma, |\gamma| \leq n} f^2(\gamma),$$

i.e., each element in the ball of radius n is counted with weight f^2.

Let us compute the generalized growth function in a particular case. Let P be the simple random walk operator on the free group with the standard set of generators of cardinality k as in Section 4.1. Let g be the unique radial eigenfunction of P corresponding to the eigenvalue $\|P\|$ and such that $g(id) = 1$. Explicitly we have:

$$g(\gamma) = \left(\frac{k-2}{k}|\gamma| + 1\right)\left(\frac{1}{\sqrt{k-1}}\right)^{|\gamma|}.$$

Then we have

$$\sum_{\gamma \in \Gamma, |\gamma| \leq n} g^2(\gamma) =$$

$$n^3\left(\frac{k^2 - 4k + 4}{3k^2 - 3k}\right) + n^2\left(\frac{3k^2 - 8k + 4}{2k^2 - 2k}\right) + n\left(\frac{7k^2 - 16k + 4}{6k^2 - 6k}\right) + 1.$$

This shows that the generalized growth is like n^3. In particular the sequence of balls is a generalized Følner sequence.

By analogy to (5) we conjecture that the fact that the generalized growth function for the free groups is like n^3 explains that for the free groups one has (see [16]):

$$c^{-1}\lambda^{2n} n^{-\frac{3}{2}} \leq P^{2n}(id, id) \leq c\lambda^{2n} n^{-\frac{3}{2}},$$

where c is a constant and λ is the norm of P.

4.2 Free products of finite groups

Random walks on free products of finite groups were already considered in [1], [3], [17] and [21].

Let us consider the group $\mathbf{Z}_m \star \mathbf{Z}_n$ with the following generating set:

- if $m \neq 2$ we take $\{\pm 1\}$ as generators of $\mathbf{Z}_m = \{0, 1, \dots, m-1\}$;
- we take $\{1\}$ as a generator of $\mathbf{Z}_2 = \{0, 1\}$.

In Figure 2 we represent the Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the above set of generators. In general the Cayley graph for $\mathbf{Z}_m \star \mathbf{Z}_n$ with the generating set defined above has the following construction:

- m-gons and n-gons are attached to each other;
- at each vertex of an n-gon there is one m-gon attached and at each vertex of an m-gon there is one n-gon attached.

4.2.1 $\mathbf{Z}_2 \star \mathbf{Z}_4$

We will present our method in the special case for $\mathbf{Z}_2 \star \mathbf{Z}_4$. The Cayley graph for this group is represented in Figure 2. Our aim is to construct the eigenfunction f of the random walk operator satisfying the generalized Følner condition. By Theorem 3, the eigenvalue corresponding to this eigenfunction is equal to the norm of a random walk operator. We will construct f in two steps.

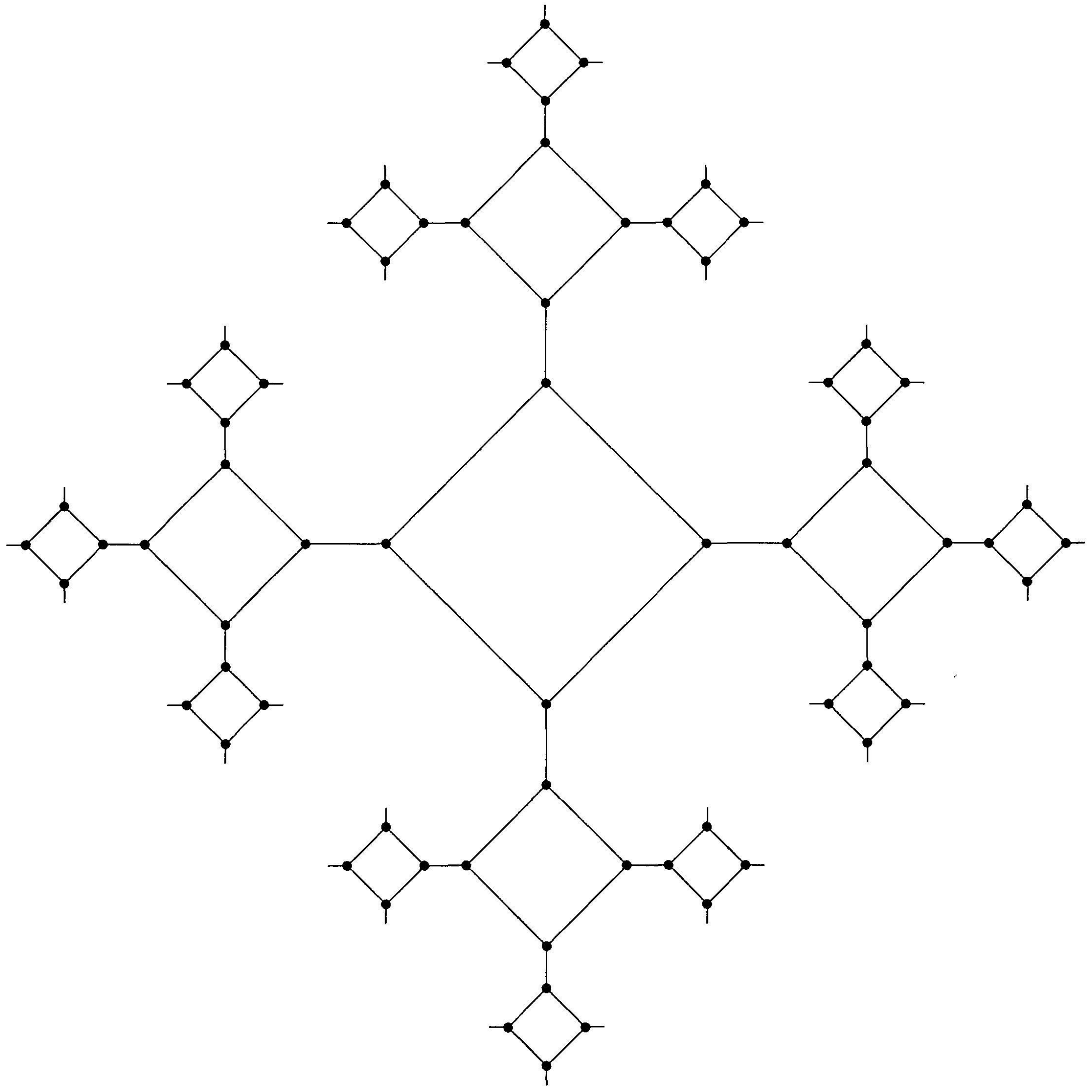

FIGURE 2
Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$

STEP 1. If we contract the squares to points, the Cayley graph for $\mathbf{Z}_2 \star \mathbf{Z}_4$ is deformed to the homogeneous tree T_4 of order 4 (each vertex has 4 neighbors), which is represented in Figure 3. First of all we construct a function on vertices of T_4 satisfying the generalized Følner condition.

We draw the graph T_4 as in Figure 3, i.e. with one point set apart at infinity. The level lines or horocycles are marked by dotted lines. Each vertex of T_4 has one neighbor above and three neighbors below.

Let us fix two positive numbers r, s and define the positive function g on the vertices of the tree T_4

$$g\colon (\text{vertices of } T_4) \to \mathbf{R}_+$$

as follows:

if w is a neighbor of v lying below v then (see Figure 4)

(1) $g(w) = rg(v)$ if w is the right or left neighbor;

(2) $g(w) = sg(v)$ if w is the middle neighbor.

The above defines the function g up to a constant. Let us fix one vertex e (for instance lying on the horocycle of level 0) and put $g(e) = 1$.

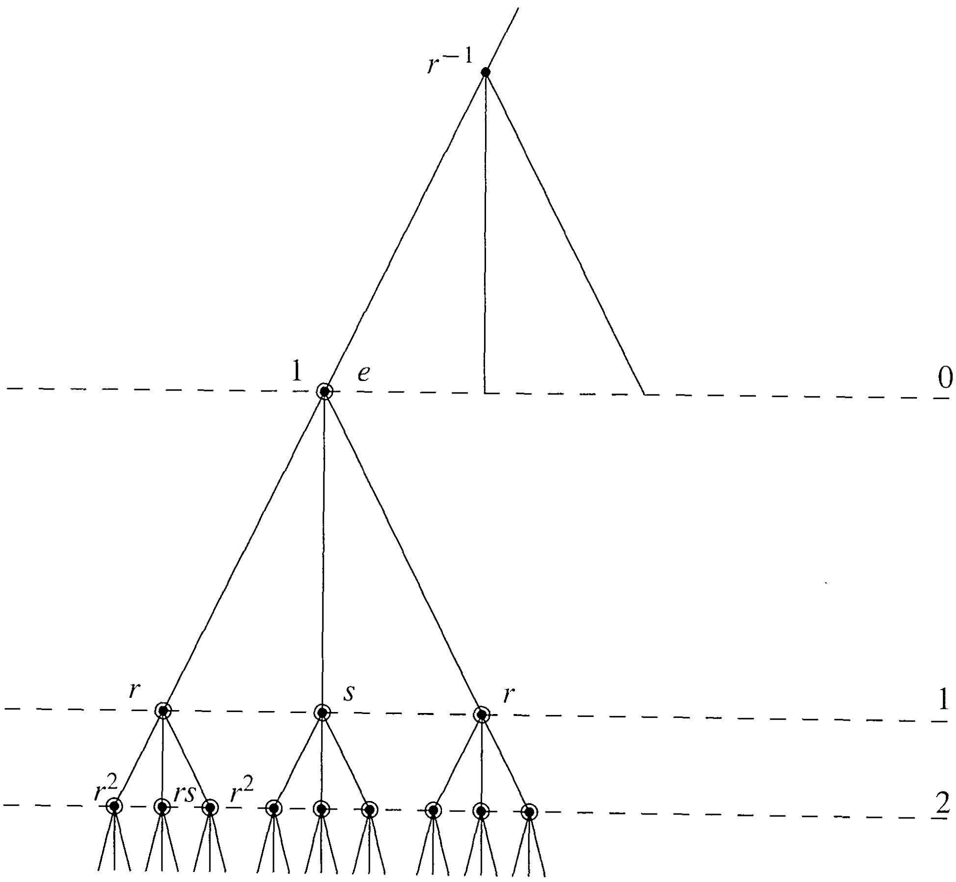

FIGURE 3

Tree T_4 of order 4

Now we need

LEMMA 4. *For* $2r^2 + s^2 = 1$ *the function* g *satisfies the generalized Følner condition, i.e. there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* T_4 *such that*

$$\frac{\sum_{v \in \partial A_n} g^2(v)}{\sum_{v \in A_n} g^2(v)} \rightarrow_{n \rightarrow \infty} 0\,.$$

Proof. Let A_n be the subset of vertices of the tree T_4 consisting of e and the vertices lying below e up to the level n (in Figure 3 the vertices of A_2 are marked with circles).

One can easily see that

$$\sum_{v \in A_n} g^2(v) = n + 1\,,$$

$$\sum_{v \in \partial A_n} g^2(v) = 2\,.$$

Thus $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence for P corresponding to g. □

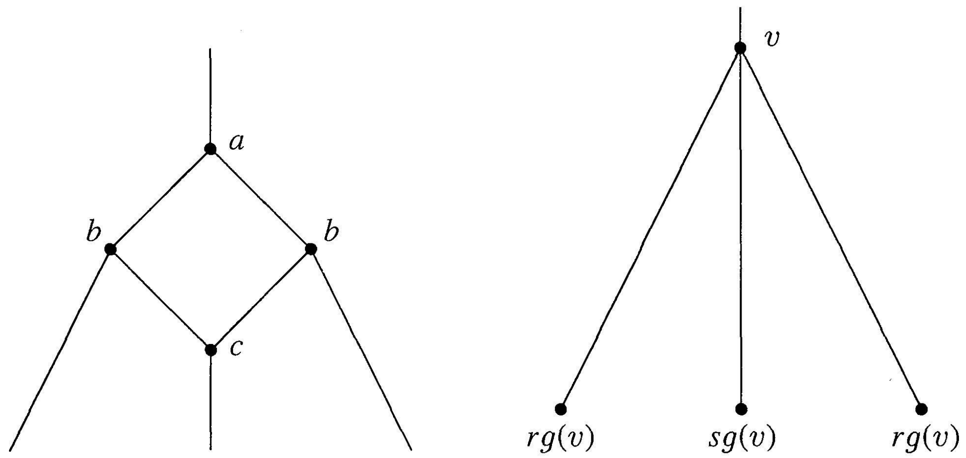

FIGURE 4
Labelling of vertices and the definition of the function g

STEP 2. The second step consists of labelling the vertices of the Cayley graph of $\mathbf{Z}_2 \star \mathbf{Z}_4$ with a, b or c (the precise values of the numbers a, b and c are given later). The vertices of each square are labelled as in Figure 4. This defines the unique labelling if we bear in mind the way we have drawn the tree T_4 obtained by contracting the squares (see Figure 3).

Now we can define the positive function f on $\mathbf{Z}_2 \star \mathbf{Z}_4$ as follows. If v is the vertex of type t ($t = a$, b or c) of the square which corresponds to the vertex w of the tree T_4 then

$$f(v) = tg(w).$$

We want to find a, b, c, r, s and λ so that f is an eigenfunction of the random walk operator P with the eigenvalue λ.

Let us write the equation

$$Pf = \lambda f$$

for vertices of type a, b and c. On a vertex of type a, the function f has to satisfy the following

$$\frac{b + 2br}{3} = \lambda ar, \tag{6}$$

$$\frac{c + 2bs}{3} = \lambda as. \tag{7}$$

For a vertex of type b, function f has to satisfy

$$\frac{a + c + ar}{3} = \lambda b \tag{8}$$

and for a vertex of type c, function f has to satisfy

$$\frac{2b + as}{3} = \lambda c. \tag{9}$$

If f satisfies the above conditions it is an eigenfunction of P with the eigenvalue λ. For $2r^2 + s^2 = 1$, by Lemma 4 the function g satisfies the generalized Følner condition and so does f. So we want to have a condition

$$2r^2 + s^2 = 1. \tag{10}$$

After solving equations (6)-(10) we obtain the following values for a, b, c, r, s and λ (a, b and c are determined up to a constant so we suppose a=1):

$$a = 1; \qquad b = \frac{u\sqrt{1 - 2u^2}}{-1 + 4u^2}; \qquad c = \frac{1 - 2u^2}{-1 + 4u^2};$$

$$r = u; \qquad s = \sqrt{1 - 2u^2}; \qquad \lambda = \frac{-1 + 2u + 4u^2}{3\sqrt{1 - 2u^2}};$$

where

$$u = \frac{\sqrt{33} - 1}{8}.$$

For the above values, f is an eigenfunction of the operator P and satisfies the generalized Følner condition. By Theorem 3 the norm of the random walk operator on $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the generating subset as defined before is then equal to

$$\|P\| = \frac{\sqrt{33}+7}{\sqrt{\sqrt{33}-1}} \approx 0.98 .$$

4.2.2 General case

The idea presented for $\mathbf{Z}_2 \star \mathbf{Z}_4$ can be used in the general case for $\mathbf{Z}_n \star \mathbf{Z}_m$. As the solution involves roots of some polynomial of degree nm, we will not give details.

4.3 Mean operator on the hyperbolic plane

Let us consider the hyperbolic upper half-plane $H = \{z = x + iy \in \mathbf{C}\,;\ x \in \mathbf{R},\ y > 0\}$ with a Riemannian metric $d_H z = \frac{\sqrt{dx^2+dy^2}}{y}$ which gives rise to the measure $\mu_H = \frac{dxdy}{y^2}$. We consider the operator P,

$$Pf(z_0) = \int_{|z-z_0|=R} f(z)\, dm_R(z) ,$$

where dm_R is a uniform probability measure on a hyperbolic circle of radius R. We want to compute the norm of the operator P acting on $L^2(H, d_H z)$.

First of all let us remark that the function:

$$f(z) = \sqrt{\mathrm{Im(z)}}, \tag{11}$$

is an eigenfunction of P. An easy way to see this is to note that P commutes with isometries of H and that the isometries consisting of horizontal translations and homotheties act transitively on H. The effect of these on the function f is that they just multiply it by a constant.

Now we would like to show that one can find a Følner sequence with respect to the function f. Let us consider a sequence $\{A_n\}_{n=1}^{\infty}$ of rectangles (in the Euclidean sense) in H:

$$A_n = \{z \in H\ ;\ e^{-n} \le \mathrm{Im}(z) \le 1,\ 0 \le \mathrm{Re}(z) \le n\} .$$

It is easy to see that the measure $|\partial A_n|$ of the boundary of A_n is bounded by the measure of the following set B_n (see Figure 5):