4.3 Mean operator on the hyperbolic plane

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For the above values, f is an eigenfunction of the operator P and satisfies the generalized Følner condition. By Theorem 3 the norm of the random walk operator on $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the generating subset as defined before is then equal to

$$\|P\| = \frac{\sqrt{33}+7}{\sqrt{\sqrt{33}-1}} \approx 0.98\,.$$

4.2.2 General case

The idea presented for $\mathbf{Z}_2 \star \mathbf{Z}_4$ can be used in the general case for $\mathbf{Z}_n \star \mathbf{Z}_m$. As the solution involves roots of some polynomial of degree nm, we will not give details.

4.3 Mean operator on the hyperbolic plane

Let us consider the hyperbolic upper half-plane $H = \{z = x + iy \in \mathbf{C}\,;\ x \in \mathbf{R}\,,\ y > 0\}$ with a Riemannian metric $d_H z = \frac{\sqrt{dx^2+dy^2}}{y}$ which gives rise to the measure $\mu_H = \frac{dxdy}{y^2}$. We consider the operator P,

$$Pf(z_0) = \int_{|z-z_0|=R} f(z)\, dm_R(z)\,,$$

where dm_R is a uniform probability measure on a hyperbolic circle of radius R. We want to compute the norm of the operator P acting on $L^2(H, d_H z)$.

First of all let us remark that the function:

$$f(z) = \sqrt{\mathrm{Im(z)}}, \tag{11}$$

is an eigenfunction of P. An easy way to see this is to note that P commutes with isometries of H and that the isometries consisting of horizontal translations and homotheties act transitively on H. The effect of these on the function f is that they just multiply it by a constant.

Now we would like to show that one can find a Følner sequence with respect to the function f. Let us consider a sequence $\{A_n\}_{n=1}^{\infty}$ of rectangles (in the Euclidean sense) in H:

$$A_n = \{z \in H\,;\ e^{-n} \le \mathrm{Im}(z) \le 1,\ 0 \le \mathrm{Re}(z) \le n\}\,.$$

It is easy to see that the measure $|\partial A_n|$ of the boundary of A_n is bounded by the measure of the following set B_n (see Figure 5):

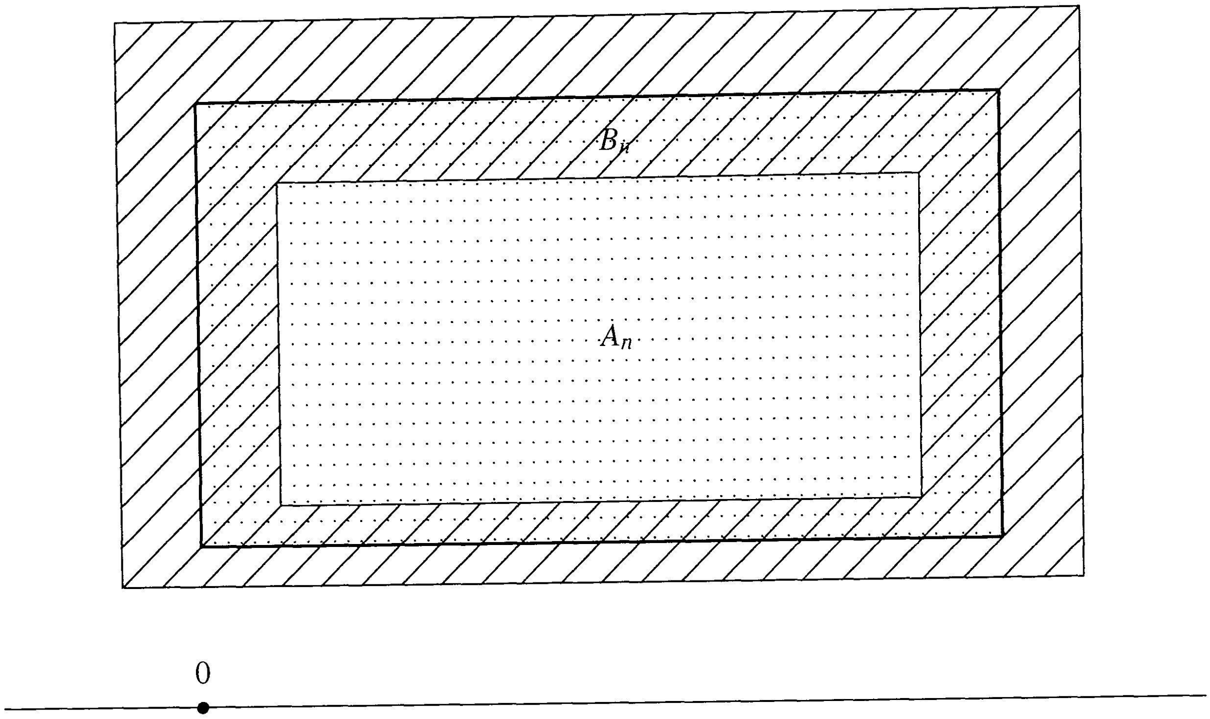

FIGURE 5
Sets A_n and B_n

$$\begin{aligned} B_n = & \{z \in H;\ -R \le \operatorname{Re}(z) \le R,\ e^R \ge \operatorname{Im}(z) \ge e^{-n-R}\} \\ & \cup \{z \in H;\ -R+n \le \operatorname{Re}(z) \le n+R,\ e^R \ge \operatorname{Im}(z) \ge e^{-n-R}\} \\ & \cup \{z \in H;\ -R \le \operatorname{Re}(z) \le n+R,\ e^R \ge \operatorname{Im}(z) \ge e^{-R}\} \\ & \cup \{z \in H;\ -R \le \operatorname{Re}(z) \le n+R,\ e^{-n+R} \ge \operatorname{Im}(z) \ge e^{-n-R}\}. \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence. Thus

$$\|P\|_{L^2(H,d_H z)\to L^2(H,d_H z)} = \int_{|z-i|=R} \sqrt{\operatorname{Im(z)}}\, dm_R(z).$$

4.4 WREATH PRODUCTS

Let G and F be finitely generated groups. We define the wreath product $G \wr F$ of these groups as follows. Elements of $G \wr F$ are couples (g, γ_1) where $g\colon F \to G$ is a function such that $g(\gamma)$ is different from the identity element id_G of G only for finitely many elements γ in F, and where γ_1 is an element of F. The multiplication in $G \wr F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where