2. PROOF OF THE THEOREM

THEOREM. *Let n be a positive, squarefree integer with either $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and with $(a, 2n) = 1$, and let j be a positive integer with $(j, 2n) = 1$ and $1 \le j \le n$. Then if $\left(\frac{-4n}{j}\right) = +1$, we have*

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\left[\frac{4in}{j}\right]+1}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right),$$

and if $\left(\frac{-4n}{j}\right) = -1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\left[\frac{(4i-2)n}{j}\right]+1}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right).$$

If $j = 1$, the result is due to Dirichlet [3], [4]. We illustrate the theorem when $n = 13$ and $j = 3$. Then $\left(\frac{-52}{3}\right) = \left(\frac{-1}{3}\right) = -1$. Thus

$$h(-13) = \frac{1}{2} \sum_{a=9}^{17} \left(\frac{-52}{a}\right).$$

Now $\left(\frac{-52}{9}\right) = \left(\frac{-52}{11}\right) = \left(\frac{-52}{15}\right) = \left(\frac{-52}{17}\right) = +1$, and so $h(-13) = \frac{1}{2}(4) = 2$. The study of class numbers relating values of the Jacobi symbol $\left(\frac{a}{n}\right)$ to $h(-n)$ when $n \equiv 3 \pmod 4$ in subintervals other than $(0, \frac{n}{2})$ has been given by numerous authors. These include among others, Berndt [1], Berndt and Chowla [2], Dirichlet [3]–[4], Holden [5]–[11], Hudson and Williams [12], Johnson and Mitchell [13], Karpinski [14], and Lerch [15]–[16]. A partial summary of these results appears in [12].

2. PROOF OF THE THEOREM

We first note that j is an odd, positive integer with $(j, n) = 1$. We write

$$\sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = \sum_{r=0}^{j-1} S_r$$

where

$$S_r = \sum_{\substack{a=1 \\ a\equiv r \pmod{j} \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right).$$

If $1 \le r \le j-1$, then there exists a unique integer k such that $1 \le k \le j-1$ and $2kn \equiv r \pmod{j}$ because $(j,n) = 1$. If $a \equiv r \pmod{j}$ with $1 \le a \le 2n-1$ and $(a,2n) = 1$, then we observe that $2kn - a \equiv 0 \pmod{j}$. Now

$$\left(\frac{-4n}{a}\right) = \left(\frac{-4n}{2kn-a}\right)$$

if k is odd, and

$$\left(\frac{-4n}{a}\right) = -\left(\frac{-4n}{2kn-a}\right)$$

if k is even. Thus,

$$\begin{aligned} \underset{1\le r\le j-1}{S_r} &= \sum_{\substack{a=(2k-2)n \\ a\equiv 0 \pmod{j} \\ (a,2n)=1}}^{2kn} \left(\frac{-4n}{2kn-a}\right) \\ &= \pm \sum_{\substack{a=(2k-2)n \\ a\equiv 0 \pmod{j} \\ (a,2n)=1}}^{2kn} \left(\frac{-4n}{a}\right) \\ &= \pm \left(\frac{-4n}{j}\right) \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right) \end{aligned}$$

where the plus sign holds if k is odd and the minus sign holds if k is even. Thus we have for each j,

$$\begin{aligned} 0 = &\left(\frac{-4n}{j}\right) \sum_{\substack{k=1 \\ (k,2)=2}}^{j} \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right) \\ &+ \left(\frac{-4n}{j}\right) \sum_{\substack{k=1 \\ (k,2)=1}}^{j} \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right) - \sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right). \end{aligned}$$

It then follows that

$$0 = -2 \sum_{\substack{k=1 \\ (k,2)=2}}^{j} \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right)$$

if $\left(\frac{-4n}{j}\right) = +1$, and

$$0 = -2 \sum_{\substack{k=1 \\ (k,2)=1}}^{j} \sum_{\substack{a=\left[\frac{(2k-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{2kn}{j}\right]} \left(\frac{-4n}{a}\right)$$

if $\left(\frac{-4n}{j}\right) = -1$. In the case that $\left(\frac{-4n}{j}\right) = +1$, we are only considering those k which are even, and so we may write $k = 2i$. In the case that $\left(\frac{-4n}{j}\right) = -1$, we are only considering those k which are odd, and so we may write $k = 2i+1$.

Thus we have proven that for each j,

$$0 = \sum_{i=1}^{\left[\frac{j}{2}\right]} \sum_{\substack{a=\left[\frac{(4i-2)n}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right)$$

if $\left(\frac{-4n}{j}\right) = +1$, and

$$0 = \sum_{i=0}^{\left[\frac{j}{2}\right]} \sum_{\substack{a=\left[\frac{4in}{j}\right]+1 \\ (a,2n)=1}}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right)$$

if $\left(\frac{-4n}{j}\right) = -1$. These subintervals clearly cover $[1, 2n-1]$ and are non-overlapping. Now Dirichlet [3], [4] showed that

$$\sum_{\substack{a=1 \\ (a,2n)=1}}^{2n-1} \left(\frac{-4n}{a}\right) = 2h(-n).$$

It follows at once that

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} \sum_{a=\left[\frac{4in}{j}\right]+1}^{\left[\frac{(4i+2)n}{j}\right]} \left(\frac{-4n}{a}\right)$$

if $\left(\frac{-4n}{j}\right) = +1$, and

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} \sum_{a=\left[\frac{(4i-2)n}{j}\right]+1}^{\left[\frac{4in}{j}\right]} \left(\frac{-4n}{a}\right)$$

if $\left(\frac{-4n}{j}\right) = -1$.

3. Remarks

In Bruce Berndt's paper "Classical Theorems on Quadratic Residues" [1], he uses the following notation:

$$S_{ji} = \sum_{\frac{(i-1)k}{j} < n < \frac{ik}{j}} \chi(n) .$$

Using this notation, we can rewrite the class number formulae as follows:

1. If $\left(\frac{-4n}{j}\right) = +1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=0}^{\frac{j-1}{2}} S_{j,2i+1} .$$

2. If $\left(\frac{-4n}{j}\right) = -1$, then we have

$$h(-n) = \frac{1}{2} \sum_{i=1}^{\frac{j-1}{2}} S_{j,2i} .$$

4. Acknowledgements

We would like to thank Zekeriya Tufekci at Clemson University for numerical data which assisted us in obtaining the results in this paper.