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A LOCAL-GLOBAL PRINCIPLE FOR NORMS FROM CYCLIC EXTENSIONS OF $\mathbf{Q}(t)$ (A DIRECT, CONSTRUCTIVE AND QUANTITATIVE APPROACH)

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ABSTRACT. Let L be a cyclic extension of $\mathbf{Q}(t)$, regular over $\mathbf{Q}$. We are concerned with the representability of a rational function $f \in \mathbf{Q}(t)$ as a norm $N^L_{\mathbf{Q}(t)}(g)$ where $g \in L$. This problem was treated by Davenport-Lewis-Schinzel in the special case $[L : \mathbf{Q}(t)] = 2$. They obtained a kind of local-global principle by proving that f is representable in the required way if, for a suitable set of integers n, $f(n)$ is likewise representable as a value of the norm-form specialized at $t = n$. In case $[L : \mathbf{Q}(t)]$ is arbitrary, it does not seem easy to extend their arguments, but a similar conclusion is a corollary of certain results on specializations of Brauer groups, obtained independently by several authors. Here we treat the general case by means of a direct method, which is self-contained as far as cohomology is concerned. Moreover our arguments are constructive and allow one to decide about the above-mentioned representability and to produce solutions when they exist. As in previous work by Serre, the method yields quantitative estimates, via sieve inequalities. We also discuss several other relevant questions.

1. INTRODUCTION

The well-known Hasse local-global principle for a cyclic extension L/K of number fields (see e.g. [CF, p. 185]) asserts that an element $a \in K^*$ is a norm from L^* (i.e. of the form $N^L_K(b)$ for some $b \in L^*$) if and only if for every place v of K and some (= all) place(s) w of L, with $w|v$, we have $a \in N^{L_w}_{K_v}(L^*_w)$ (where the subscripts denote completions). Actually Hasse's theorem holds more generally when L/K is any cyclic extension of *global fields*; these are either number fields or function fields of curves over finite fields, namely finite extensions of some field $\mathbf{F}_q(t)$.

It seems natural to investigate what happens when L/K is a cyclic extension of function fields of curves over number fields. Let us concentrate on the case when the base is rational, namely $K = k(t)$, where k is a number field.