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**Artikel:** LOCAL-GLOBAL PRINCIPLE FOR NORMS FROM CYCLIC EXTENSIONS OF  $\mathbb{Q}(t)$  (A DIRECT, CONSTRUCTIVE AND QUANTITATIVE APPROACH)  
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$$N(r, b_1, \dots, b_d) = f(r)\mu$$

where  $\mu \in \mathbf{Q}_v$  is very close to 1; in fact  $f(r)$  is near to  $f(a_v)$ , which is nonzero. By the previous remarks,  $\mu^{-1}$  is in the image of  $N(r, x_1, \dots, x_d)$  on  $\mathbf{Q}_v^d$ , hence the same must be true for  $f(r)$ , by the basic multiplicative identity for  $N$ . In particular  $f(r)$  will be a norm from  $\mathbf{Q}_v(P_r)$  to  $\mathbf{Q}_v$ .

Let now  $S$  consist of the elements of  $\mathbf{Q}$  which are not poles or zeros of  $f$ , which satisfy  $[\mathbf{Q}(P_s) : \mathbf{Q}] = d$  and which are sufficiently close (in the mentioned sense) to  $a_v$ , for each  $v \in \Sigma$ . We have proved that  $f(s)$  is a norm from  $\mathbf{Q}_v(P_s)$ , for all  $s \in S$  and for all places  $v$ . By Hasse's theorem,  $f(s)$  is a norm from  $\mathbf{Q}(P_s)$ , so  $S \subset N_f$ . On the other hand  $S \cap \mathbf{Z}$  contains the complement of a thin set in an arithmetic progression, whence  $N_f$  cannot satisfy the conclusion of the Theorem (or of Corollary 1), as required.  $\square$

#### 4. AN EXAMPLE FOR THE NON-CYCLIC CASE

We show that assuming that  $L/K$  is cyclic is essential in the Theorem (as in the number-field case, as shown in [CF, Ex. 5]).

To describe a counterexample, define  $L = \mathbf{Q}(t, \sqrt{4t+3}, \sqrt{4t+7})$ ,  $f(t) = t^2$ . We proceed to show that  $\mathbf{N} \subset N_f$ . We have to show that for all large integers  $n$ ,  $n^2$  is a norm from  $L(n) := \mathbf{Q}(\sqrt{4n+3}, \sqrt{4n+7})$ . By [CF, Ex. 5.1 and 5.2, p.360] it is sufficient to show that the local degree  $[L(n)_w : \mathbf{Q}_p]$  is 4 for some prime  $p$ . Observe that the Jacobi symbol  $\left(\frac{4n+3}{4n+7}\right) = \left(\frac{-1}{4n+7}\right) = -1$ . Hence there exists some prime  $p$  dividing  $4n+7$  with an odd multiplicity and such that  $\left(\frac{4n+3}{p}\right) = -1$ . Then  $p$  ramifies in  $L(n)$  and the residual degree is 2, proving the claim. Observe that the first conclusion of Corollary 1 does not hold for  $N_f$ .

On the other hand,  $t^2$  is not a norm from  $L$  to  $K$ . Otherwise by [CF, Ex. 5.1] we could write  $t$  as the product of three norms from the three quadratic subfields of  $L$ . In other words we could write nontrivially

$$q^2(t)t = (a_1^2(t) - (4t+3)b_1^2(t))(a_2^2(t) - (4t+7)b_2^2(t))(a_3^2(t) - (4t+3)(4t+7)b_3^2(t)),$$

where  $q, a_i, b_j \in \mathbf{Q}[t]$ . We may suppose that  $a_i$  and  $b_i$  are coprime for each  $i$ , otherwise we can divide out a common factor. Now, putting  $t = 0$  we get a contradiction.