2. Examples

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.07.2024**

2. Examples

In this section we show how examples of cubical chamber complexes of nonpositive curvature arise naturally in many constructions. We indicate the features of the constructions which lead to foldability of the universal covers of the resulting complexes.

Barycentric subdivision

Recall that if P is a cell then the barycentric subdivision P' of P is the following simplicial complex: k-simplices of P' correspond to sequences $F_0 \subset F_1 \subset \cdots \subset F_k$ of faces of P, where $F_{i-1} \neq F_i$ for $i = 1, \ldots, k$, and the relation of being a face corresponds to being a subsequence. One can realize the subdivision P' geometrically inside P as follows: For each face F of P choose a point p_F in the interior of F (if F is a vertex then $p_F = F$). Then the simplex corresponding to a sequence $F_0 \subset F_1 \subset \cdots \subset F_k$ is identified with the convex hull of the set $\{p_{F_0}, p_{F_1}, \ldots, p_{F_k}\}$ in P.

For a cell complex X the barycentric subdivision X' of X is the simplicial complex whose k-simplices correspond to sequences $C_0 \subset C_1 \subset \cdots \subset C_k$ of cells of X, with $C_{i-1} \neq C_i$ for $i = 1, \ldots, k$. This corresponds to subdividing barycentrically all the cells of X in a consistent way.

We recall some well known facts related to barycentric subdivision.

Lemma 2.1. *The barycentric subdivision X' of a cell complex X is a foldable flag complex.*

Proof. We note that a folding of X' onto the simplex spanned by the set $\{0, 1, \ldots, \dim X\}$ is well defined by assigning to each vertex in X' the dimension of the corresponding cell in X.

If A is a set of vertices of X' pairwise connected by edges, then the set of corresponding cells of X can be ordered by inclusion. But this means that A spans the simplex in X' corresponding to this ordered sequence. Thus X' is a flag complex. □

Lemma 2.2. *Let v be a vertex of a cell complex X. Then the complexes $(X')_v$ and $(X_v)'$ are isomorphic.*

Proof. Simplices in both $(X')_v$ and $(X_v)'$ correspond to sequences $C_0 \subset C_1 \subset \cdots \subset C_k$ of cells of X containing v and distinct from v. □

LEMMA 2.3. *Let v be a vertex of a cell P. Then the complex $(P')_v$ is isomorphic to the complex $[(\partial P)' \times I]_{v\times 1}$, where I is a 1-cell with a vertex 1, and $\times$ denotes the product of cell complexes.*

Proof. Clearly, both complexes are isomorphic to the simplicial cone over the complex $(\partial P)'_v$. □

HYPERBOLIZATIONS

We briefly describe two procedures which turn cell complexes into nonpositively curved cubical complexes (for more details, see [DJ1], [CD]). We also discuss when the resulting complexes are foldable chamber complexes.

THE PRODUCT WITH INTERVAL PROCEDURE. Define a functor h_1 from the category of cell complexes to the category of cubical complexes, inductively with respect to the dimension of initial cell complexes. Let K be a cell complex. If $\dim K \leq 1$, set $h_1(K) = K$. Now consider a cell complex K with $\dim K = i+1$. Assuming inductively that h_1 has been already defined for all cell complexes of dimension $\leq i$, define a hyperbolized complex $h_1(K)$ as follows. Glue "hyperbolized $(i+1)$-cell" $h_1(C) := h_1(\partial C) \times [-1,1]$ (corresponding to $(i+1)$-cells C of K) to the complex $h_1(K^{(i)}) \times \{-1,1\}$ according to the identifications of the two copies of the sets $h_1(\partial C) \times \{-1,1\}$, one in $h_1(C)$ and second in $h_1(K^{(i)}) \times \{-1,1\}$, by the identity maps. Note that $h_1(\partial C)$ is identified with a subset of $h_1(K^{(i)})$ by the (inductively verified) functoriality of h_1 for cell complexes of dimension $\leq i$.

THE MÖBIUS BAND PROCEDURE. Define a functor h_2 on the category of cubical complexes, inductively with respect to the dimension of initial complexes. Let K be a cubical complex. If $\dim K \leq 1$, set $h_2(K) = K$. Now consider a cubical complex K with $\dim K = i+1$. Assuming inductively that h_2 has been already defined for all cubical complexes of dimension $\leq i$, define a hyperbolized complex $h_2(K)$ as follows. For each $(i+1)$-cell $C \equiv [-1,1]^{i+1}$ in K put $h_2(C) = (h_2(\partial C) \times [-1,1])/\tau$. Here τ is the involution $\tau(x,t) = (a(x), -t)$ on $h_2(\partial C) \times [-1,1]$, where a is the combinatorial automorphism of $h_2(\partial C)$ induced from the antipodal automorphism of ∂C by the (inductively verified) functoriality of h_2 for cubical complexes of dimension $\leq i$. Identify $h_2(\partial C)$ with the image of $h_2(\partial C) \times 1$ in $h_2(C)$, and then glue each $h_2(C)$ to $h_2(K^{(i)})$ along $h_2(\partial C)$ using the identity map.

Note that for both procedures above each vertex in the hyperbolized complex $h_j(K)$ corresponds to a unique vertex in the initial complex K. The next lemma shows how the links at such vertices in the corresponding complexes are related.

LEMMA 2.4. *Let v be a vertex of $h_j(K)$ and v_0 the corresponding vertex of K. Then the links $(h_j(K))_v$ and $(K')_{v_0}$ are isomorphic.*

Proof. We proceed by induction with respect to dimension of K. Clearly, if $\dim K = 1$ then $v_0 = v$, $h_j(K) = K$ and $K_v \cong (K')_v$. Let $\dim K = n \geq 2$. Denote by $\overline{v}$ the vertex in $h_j(K^{(n-1)})$ corresponding to v. Then by the inductive hypothesis, the links $[h_j(K^{(n-1)})]_{\overline{v}}$ and $[(K^{(n-1)})']_{v_0}$ are isomorphic. On the other hand, it follows from the descriptions of the procedures h_j that for each hyperbolized n-cell $h_j(C)$ containing v we have $[h_j(C)]_v \cong [h_j(\partial C) \times I]_{\overline{v} \times 1}$. The lemma follows then from Lemma 2.3. □

PROPOSITION 2.5. *Let K be a cell (respectively cubical) chamber complex which is locally gallery connected. Then the hyperbolized complex $h_1(K)$ (respectively $h_2(K)$) is a nonpositively curved cubical chamber complex. Moreover, its universal cover, with the induced cubical structure, is foldable.*

Proof. It is clear from the construction that $h_j(K)$ is a cubical chamber complex. By Lemmas 2.2 and 2.4, we have $[h_j(K)]_v \cong (K_{v_0})'$. It follows from Lemma 2.1 that the links of $h_j(K)$, at all vertices, are foldable flag complexes. Now Gromov's Lemma 1.5 implies that the complexes $h_j(K)$ are nonpositively curved. It is immediate from Lemma 2.4 that $h_j(K)$ is locally gallery connected if K is. The last part of Proposition 2.5 follows then from Lemma 1.1. □

Note that if X is the universal cover of a hyperbolized complex $h_j(K)$ as in the above Proposition, then the fundamental group $\pi_1(h_j(K))$ acts on X freely by automorphisms. The complex X and the group Γ are then examples of a complex and a group as in Theorems 1 and 2 of the introduction.

The universal cover of $h_j(K)$ is hyperbolic in many cases, but not always, see [Gr], [CD].

ZONOTOPAL COMPLEXES

In this subsection we briefly describe an extended class of cell complexes to which the Möbius band hyperbolization procedure can be applied, see [DJS] for more details. (Recall that the product with interval procedure applies to all cell complexes.)

An *arrangement* in a real vector space V is a finite collection $\mathcal{H}$ of linear subspaces of V with codimension one. Elements of $\mathcal{H}$ are called *hyperplanes*. An arrangement $\mathcal{H}$ is *essential* if the intersection $\bigcap \mathcal{H}$ of all hyperplanes in $\mathcal{H}$ is $\{0\}$.

Let $\mathcal{H}$ be an essential arrangement. For each hyperplane $H \in \mathcal{H}$ consider a linear functional $f_H \in V^*$ with $\ker f_H = H$. Denote by $Z_{\mathcal{H}}$ the convex polytope in V^* which is the convex hull of the set

$$\Big\{ \sum_{H \in \mathcal{H}} \varepsilon_H \cdot f_H \mid \varepsilon_H = \pm 1 \Big\}.$$

It turns out that the combinatorial structure of $Z_{\mathcal{H}}$ does not depend on the choice of the functionals f_H. In fact the polytope $Z_{\mathcal{H}}$ is dual to the arrangement $\mathcal{H}$ in the sense that its boundary is dual to the spherical cell complex determined by the intersection of $\mathcal{H}$ with the unit sphere in V.

Polytopes of the form $Z_{\mathcal{H}}$ as above are called *zonotopes* (see [B-Z]). A cell complex is *zonotopal* if all of its cells are zonotopes. The boundary of a zonotope is an example of a zonotopal complex, since each face of a zonotope is a zonotope.

The important feature of a zonotope $Z = Z_{\mathcal{H}}$ is that the central symmetry $f \mapsto -f$ of V^* induces a combinatorial antipodal automorphism of Z. This allows to apply the Möbius band hyperbolization h_2 to zonotopal complexes. By the same arguments as in the previous subsection we get the following result.

PROPOSITION 2.6. *Let K be a zonotopal chamber complex which is locally gallery connected. Then the hyperbolized complex $h_2(K)$ is a nonpositively curved cubical chamber complex. Moreover, the universal cover of $h_2(K)$, with the induced cubical structure, is foldable.*

BLOW-UPS OF ARRANGEMENTS

An arrangement $\mathcal{H}$ in a real vector space V determines an arrangement $P(\mathcal{H})$ of projective hyperplanes in the projective space $P(V)$. If $\mathcal{H}$ is essential then $P(\mathcal{H})$ divides the space $P(V)$ into convex spherical polytopes, so that it becomes a chamber complex. It is proved in [DJS] that the cell structure dual to the above converts the space $P(V)$ into a zonotopal chamber complex.

It is possible to interpret the hyperbolization procedure h_2, applied to a zonotopal complex as above, as a sort of blow-up with respect to the divisor in $P(V)$ consisting of all subspaces of codimension greater than one which are intersections of hyperplanes in $P(\mathcal{H})$, see [DJS]. By Proposition 2.6, this blow-up produces a nonpositively curved cubical chamber complex whose universal cover is foldable.

In [DJS], the procedure described above is called the maximal blow-up. In the same paper some refinements of this procedure, called partial blow-ups, are discussed. In many natural cases these partial blow-ups result in cubical chamber complexes of nonpositive curvature.

SIMPLE POLYTOPES

A convex polytope P is *simple* if the link of P at any vertex is a simplex. Equivalently, P is simple if the boundary complex $\partial\widetilde{P}$ of the dual polytope $\widetilde{P}$ is a simplicial complex.

Any n-dimensional simple polytope P can be subdivided canonically into a cubical complex $P_\square$ in such a way that vertices of $P_\square$ correspond to cells of P and each cubical n-cell of $P_\square$ is spanned by the set of vertices corresponding to cells of P containing a fixed vertex of P. See section 1.2 of [DJS] for a more detailed description of this subdivision and for the proof of the following lemma.

LEMMA 2.7. *Let v be the vertex of $P_\square$ corresponding to P. Then the link $(P_\square)_v$ is isomorphic to $\partial\widetilde{P}$.*

COROLLARY 2.8. *The following conditions are equivalent:*

(1) *$P_\square$ is foldable;*

(2) *$\partial\widetilde{P}$ is foldable;*

(3) *for each codimension 2 simplex C in $\partial\widetilde{P}$ the link $(\partial\widetilde{P})_C$ is even-gonal;*

(4) *each 2-dimensional face F of P is even-gonal.*

Proof. Conditions (1) and (2) are equivalent by Lemma 2.7. The equivalence of (2) and (3) follows from Lemma 1.2. And (3) and (4) are just the dual expressions of the same condition. □

REMARK 2.9. A polytope P satisfies Condition (4) of Corollary 2.8 if and only if P is a zonotope, see Proposition 2.2.14, p. 64, in [B-Z]. Therefore, the cubical subdivision $P_\square$ of a simple polytope P is foldable if and only if P is a (simple) zonotope.

Any face F of a simple polytope P is also simple and the cubical subdivision $P_\square$ restricted to F agrees with the subdivision $F_\square$. Let X be a *simple cell complex*, i.e. a complex all cells of which are simple. Then the canonical cubical subdivisions of the cells of X are consistent and determine a subdivision $X_\square$ of X.

From [DJS] we recall the following

LEMMA 2.10. *The canonical cubical subdivision of a simple chamber complex X is nonpositively curved if and only if the following two conditions are satisfied:*

(1) *for each chamber P of X the boundary $\partial\widetilde{P}$ of the dual simplicial polytope is a flag complex;*

(2) *for each vertex v of X the link X_v is a flag complex.*

In view of Lemma 1.1, we can summarize the considerations of this subsection in the following

PROPOSITION 2.11. *Let K be a chamber complex satisfying the following conditions:*

(1) *all cells K are simple zonotopes;*

(2) *the links of K at all vertices are gallery connected and foldable.*

Then the cubical subdivision $K_\square$ is nonpositively curved and its universal cover with the induced cubical structure is a foldable chamber complex.

POLYGONAL COMPLEXES

Recall that a 2-dimensional cell complex is called a *polygonal complex*. Polygonal complexes arise naturally in combinatorial group theory. The class of polygonal complexes is very rich, see [Bar], [BB], [BS], [Be], [Sw].

Since polygonal complexes are simple, Remark 2.9 and Proposition 2.11 imply the following assertion.

PROPOSITION 2.12. *Let K be a polygonal complex satisfying the following conditions:*

(1) *all 2-cells of K have an even number of sides;*

(2) *the links of K at all vertices are connected bipartite graphs.*

Then the cubical subdivision $K_\square$ is nonpositively curved and its universal cover with the induced cubical structure is a foldable chamber complex.

A big class of polygonal complexes is constituted by Cayley complexes of presentations of groups, on which the corresponding groups act freely by combinatorial automorphisms. Using Proposition 2.12, it is then easy to decide in terms of the presentation whether a group acting on its Cayley complex satisfies the assumptions of Theorems 1–3 of the introduction. Many other examples satisfying these assumptions can be constructed using various methods, see [BB], [BS], [Sw].

TORIC MANIFOLDS

In this subsection, we recall the construction of toric manifolds from [DJ2]. Let $\mathcal{F}$ be the set of codimension 1 faces of a simple polytope P of dimension n. A map $\lambda\colon \mathcal{F} \to (Z_2)^n$ is a *characteristic function* for P, if for every vertex v of P the set $\{\lambda(F) \mid F \in \mathcal{F}, v \in F\}$ is a basis for $(Z_2)^n$. Let $\sim$ be the equivalence relation on the set $P \times (Z_2)^n$ defined by $(x,s) \sim (x,t)$ if $x \in F$ and $s \equiv t \bmod \lambda(F)$. Put $M(P,\lambda) := P \times (Z_2)^n/\!\sim$ and note that $M(P,\lambda)$ is a simple chamber complex with chambers the images of the sets $P \times \{s\}$ in the quotient. The projection $P \times (Z_2)^n \to P$ induces a combinatorial map $\pi\colon M(P,\lambda) \to P$ which is injective on cells of $M(P,\lambda)$. By Proposition 1.7 of [DJ2], $M(P,\lambda)$ is a closed manifold and it is called a *toric manifold.*

PROPOSITION 2.13. *Let P be a simple polytope with even-gonal 2-dimensional faces and λ be a characteristic function for P. Then the standard cubical subdivision of the toric manifold $M(P,\lambda)$ is foldable and nonpositively curved.*

Proof. According to Corollary 2.8, there is a folding ϕ of $P_\Box$. Furthermore, we can view the map π as a nondegenerate combinatorial map $M(P,\lambda)_\Box \to P_\Box$. Then the composition $\phi \circ \pi$ is a folding of $M(P,\lambda)_\Box$.

Nonpositive curvature of $M(P,\lambda)_\Box$ follows from Lemma 2.10 since the links of $M(P,\lambda)$ are isomorphic to the boundaries of hyperoctahedra (simplicial polytopes dual to cubes). $\Box$

Let X be the universal cover of $M(P,\lambda)_\Box$ with the induced cubical structure. Then the fundamental group Γ of $M(P,\lambda)$ acts on X freely by combinatorial automorphisms and the pair X and Γ satisfies the assumptions of Theorems 1 and 2 of the introduction.

RIGHT ANGLED COXETER COMPLEXES

Given a simplicial complex K define the *cubical cone* C_cK to be the unique cubical complex with distinguished vertex v_0 satisfying the following properties:

(1) C_cK is the union of those cells which contain v_0 ;

(2) the link $(C_cK)_{v_0}$ is isomorphic to K.

Define the *base* B_K of this cone to be the subcomplex consisting of all cells not containing v_0. Then B_K is canonically isomorphic to the standard cubical subdivision $K_\square$ of K and hence the vertices of B_K naturally correspond to the simplices of K.

For each vertex v of K define a *coface* F_v in B_K as follows. Let v' be the vertex in B_K corresponding to v. Then F_v is a subcomplex consisting of all cells of B_K which contain v'.

Let I be a finite set and $M = [m_{ij}]$ a symmetric matrix indexed by $I \times I$. Assume that $m_{ii} = 1$ and $m_{ij} \in \{2, +\infty\}$ for all $i, j \in I$, $i \neq j$. A *right angled Coxeter group* is a group W_M given by a presentation

$$W_M = \langle s_i \mid (s_i s_j)^{m_{ij}} \rangle$$

for some matrix M as above. Any such matrix will be called a *right angled matrix*.

For a right angled matrix M define the graph Γ_M as follows. The set $\{v_i \mid i \in I\}$ of vertices of Γ_M is in 1-1 correspondence with I, and vertices v_i, v_j are connected by an edge if and only if $m_{ij} = 2$. The graph Γ_M determines uniquely a flag complex K_M with the same vertex and edge set: a set of vertices spans a simplex in K_M if and only if any two vertices in this set are connected in Γ_M by an edge.

The *Coxeter complex* of the right angled Coxeter group W_M is the quotient $\Sigma_M = W_M \times C_cK_M/\sim$ modulo the equivalence relation determined by all the equivalences $(w_1, x) \sim (w_2, x)$ with $x \in F_{v_i}$ and $w_1^{-1}w_2 = s_i$ for all $i \in I$. G. Moussong proved the following [Mo].

PROPOSITION 2.14. *The Coxeter complex of a right angled Coxeter group is a locally compact simply connected nonpositively curved cubical complex on which the group acts properly and cocompactly by combinatorial automorphisms.*

In addition to Proposition 2.14 we have the following

LEMMA 2.15. *Assume that for some right angled matrix M the complex K_M is a foldable chamber complex. Then the Coxeter complex Σ_M is also a foldable chamber complex.*

Proof. Note that C_cK_M is foldable since K_M is foldable. Let ϕ be a folding of C_cK_M and $p\colon \Sigma_M \to C_cK_M$ be the nondegenerate combinatorial map induced by the projection $W_M \times C_cK_M \to C_cK_M$. Then clearly the composition $\phi \circ p$ is a folding of Σ_M.

Observe that C_cK_M and hence Σ_M is dimensionally homogeneous since K_M is. Hence to show that Σ_M is a chamber complex, it remains to prove gallery connectedness. To that end let $[(w_1, C_1)]$ and $[(w_2, C_2)]$ be two chambers of Σ_M. By the gallery connectedness of C_cK_M — which is immediate from the gallery connectedness of K_M — it is clear that there is a gallery connecting the above chambers if $w_1 w_2^{-1} = s_i$. The existence of a connecting gallery in the general case follows by induction on the word length of $w_1 w_2^{-1}$ in W_M. □

RIGHT ANGLED BUILDINGS AND GRAPH PRODUCTS OF GROUPS

In [Da] M. Davis defines buildings of type M for a class of matrices which contains right angled matrices. If M is a right angled matrix then any building of type M is a cubical complex and its apartments are isomorphic to the Coxeter complex Σ_M. It is proved in [Da] that any such building is nonpositively curved and simply connected. Moreover, since any two cells of a building lie in a common apartment and since there is a nondegenerate combinatorial map of a building onto any of its apartments, we have the following

PROPOSITION 2.16. *Let M be a right angled matrix for which the complex K_M is a foldable chamber complex. Then any building of type M is a simply connected foldable cubical chamber complex of nonpositive curvature.*

Let M be a right angled matrix over I, and for each $i \in I$ let G_i be a group. Define the *graph product* of the groups G_i (with respect to M) as the quotient of the free product of the groups G_i, $i \in I$, by the normal subgroup generated by all commutators of the form $[g_i, g_j]$, where $g_i \in G_i$, $g_j \in G_j$ and $m_{ij} = 2$. Davis proved [Da] that the graph product of groups (with respect to M) acts cocompactly by automorphisms on a building of type M. He also showed that the building is locally compact and the action is proper if the groups G_i in the product are finite. Moreover, if the assumptions of Corollary 2.16 for M are satisfied, then the action preserves the folding of the building.