4. NONEXISTENCE OF FREE SUBGROUPS

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Proof. Let φ be an automorphism of X. If φ fixes a point p of Λ_i^*, then p can be chosen as a vertex or a midpoint of an edge. If p is a vertex, then the preimage X' of p under r_i is a closed and convex subcomplex of X. If p is the midpoint of an edge, X' is a hyperspace and as a union of walls, carries a natural cubical structure. In either case, X' is a closed, convex and φ-invariant subset of X, and therefore φ is semisimple if and only if the restriction $\varphi|_{X'}$ is semisimple. Since moreover X' is a simply connected folded cubical chamber complex of nonpositive curvature and of dimension lower than X, we can assume by induction on $\dim X$ that the action of φ on all the trees Λ_i^* is axial.

Let a_i be an axis of φ in Λ_i^* (unique up to parameter). Let $X_i = r_i^{-1}(a_i)$. Since r_i is surjective, X_i is non-empty. Furthermore, X_i is a closed, convex and φ-invariant subcomplex of X.

Set $Y_1 := X_1$. The image of Y_1 under r_2 is path connected and φ-invariant, hence contains a_2. Let $Y_2 = Y_1 \cap X_2$. Then Y_2 is non-empty, closed, convex and φ-invariant. By induction we get that $Y = X_1 \cap \ldots \cap X_n$ is a non-empty, closed, convex and φ-invariant subcomplex of X. It is then sufficient to prove semisimplicity for the restriction $\varphi|_Y$. Note that $Y = r^{-1}(F)$, where $F \cong \mathbf{R}^n$ is the flat

$$F = \{(a_1(t_1), \ldots, a_n(t_n)) \mid t_i \in \mathbf{R}\}$$

in the product of trees. Now φ operates as a translation on F, hence the displacement of φ on F is constant, say $= \delta$. Since r is injective, we can consider Y as a closed subcomplex of F, namely a union of chambers. The metric on Y is the induced path metric. It follows easily that there are only finitely many possible values for the distance in Y from a point x to its image φx, if the location of x in its chamber is given. □

4. Nonexistence of free subgroups

In this section we discuss the proof of Theorem 2 of the introduction. We assume throughout this section that X is a simply connected folded cubical chamber complex of nonpositive curvature and that $\Gamma \subset \mathrm{Aut}(X)$ is a group that preserves the folding of X (this can be always assumed by passing to a finite index normal subgroup if necessary) and does not contain a free nonabelian subgroup acting freely on X. By equivariance of the maps r_i, the same holds for the actions of Γ on the trees Λ_i^*. Up to a subgroup of index two, there are three possibilities for each particular i [PV]:

(0) Γ fixes a point of Λ_i^* ;

(1) Γ fixes no point of Λ_i^*, but precisely one end of Λ_i^* ;

(2) Γ fixes no point of Λ_i^*, but precisely two ends of Λ_i^*.

Thus by passing to a subgroup of Γ of index at most 2^n, we can assume that the above three alternatives hold for all i. Corresponding to the alternative, we say that i is an *index of type* 0, 1 or 2 respectively.

We first construct a homomorphism $h = (h_1, \dots, h_n) \colon \Gamma \to \mathbf{Z}^n$ as claimed. If Γ fixes a point of Λ_i^*, we define h_i to be the trivial homomorphism. If Γ does not fix a point of Λ_i^*, we let ω_i be the end or one of the two ends of Λ_i^* fixed by Γ. The Busemann function $b_i \colon \Lambda_i^* \to \mathbf{R}$ at ω_i is well defined up to an additive constant (see [Ba], Section 1 of Chapter II). Since Γ fixes ω_i,

$$h_i(\phi) := b_i(\phi p) - b_i(p), \quad p \in \Lambda_i^* ,$$

is a well defined homomorphism $h_i \colon \Gamma \to \mathbf{Z}$, called the *Busemann homomorphism*. Note that h_i is integer valued since Λ_i^* is a simplicial tree and Γ acts by automorphisms. This completes the definition of $h = (h_1, \dots, h_n)$. We set

$$\Delta_i = \ker h_i \quad \text{and} \quad \Delta = \bigcap \Delta_i = \ker h .$$

PROPOSITION 4.1. *Δ consists precisely of the elliptic elements of Γ.*

Proof. If the action of Γ on Λ_i^* has a fixed point, then any $\phi \in \Gamma$ is elliptic on Λ_i^* and $\Delta_i = \Gamma$. If Γ does not have a fixed point in Λ_i^*, but fixes a point $\xi_i \in \Lambda_i^*(\infty)$ and $\phi \in \Gamma$ is axial on Λ_i^*, then ξ_i is an end point of the axis of ϕ. Then $h_i(\phi) \neq 0$. Hence by Proposition 3.5, any $\phi \in \Delta$ is elliptic on X. Conversely, if $\phi \in \Gamma$ is elliptic on X, then $\phi \in \Delta$. □

For the proof of the other assertions of Theorem 2 we need some more preparations.

LEMMA 4.2. *Let Λ be a simplicial tree on which Γ acts by automorphisms. Suppose Δ fixes a point of Λ. Then either Γ fixes a point of Λ or exactly two points in $\Lambda(\infty)$.*

Proof. Since Δ is a normal subgroup of Γ, the set Φ of fixed points of Δ is Γ-invariant. Now Φ is a subtree of Λ, hence we can assume $\Phi = \Lambda$. Then the quotient action by Γ/Δ on Λ is well defined.

Suppose that Γ/Δ contains an element ϕ which is axial on Λ. Since Γ/Δ is abelian, it leaves the unique axis of ϕ invariant and fixes the endpoints of the axis.

Suppose now that all elements of Γ/Δ are elliptic on Λ. Let $\phi_1, \dots, \phi_k$ be a system of generators. The set of fixed points of ϕ_1 is a Γ/Δ-invariant subtree. Replacing Λ by this subtree, we can assume that $\phi_1 = \mathrm{id}_\Lambda$. The quotient of Γ/Δ by the subgroup generated by ϕ_1 is abelian and has a system of $k-1$ generators. Induction on k shows that Γ has a fixed point. □

If i is an index of type 0 and $p \in \Lambda_i^*$ a fixed point, then $X' := r_i^{-1}(p) \subset X$ is closed, convex and Γ-invariant. In particular, $X'(\infty) \subset X(\infty)$ is Γ-invariant. Although X' is not a subcomplex if p is not a vertex, it is parallel to the walls with label i in the chambers it intersects. Hence we obtain a natural cubical structure on X' with a folding onto an $(n-1)$-cube, and Γ preserves this cubical structure and folding. Hence by passing to such subspaces if necessary, we can assume that no indices of type 0 occur.

Let i be an index of type 2. Let $\alpha_i, \omega_i \in \Lambda_i^*(\infty)$ be the fixed points of Γ and σ_i the unit speed geodesic from α_i to ω_i. Then σ_i is Γ-invariant and $\Delta_i = \mathrm{Stab}(\sigma_i(t))$ for all $t \in \mathbf{R}$. Hence $X' = r_i^{-1}(\mathrm{im}\ \sigma_i)$ is a closed, convex and Γ-invariant subcomplex of X. Hence by passing to such subspaces if necessary, we can assume that $\Lambda_i^* = \mathrm{im}\ \sigma_i \cong \mathbf{R}$ for all indices i of type 2.

PROPOSITION 4.3. *If there are no indices of type* 1, *then there is a* Γ-*invariant convex subset* $E \subset X$ *isometric to a Euclidean space of dimension* $k \in \{0, \dots, n\}$ *and an exact sequence*

$$0 \to \Delta \to \Gamma \to \mathbf{Z}^k \to 0$$

such that Δ *fixes* E *pointwise and such that the quotient* $\Gamma/\Delta \cong \mathbf{Z}^k$ *acts on* E *as a cocompact lattice of translations.*

Proof. After reductions as above we can assume that all indices are of type 2, that $\Lambda_i^* \cong \mathbf{R}$ for all i and that Δ fixes each point of $\prod \Lambda_i^*$. Since r is an injection, Δ fixes each point of X.

The image im h of the homomorphism h is a subgroup of the group $\mathbf{Z}^n$, hence it is isomorphic to $\mathbf{Z}^k$ for some $k \leq n$. Thus we may identify the quotient group Γ/Δ with $\mathbf{Z}^k$. Consider the quotient action of $\mathbf{Z}^k = \Gamma/\Delta$ on X, which is well defined since Δ acts trivially on X. This action is free and the elements are semisimple by Proposition 3.6. Applying the Flat Torus Theorem, see [CE] and [BH], we get that there exists a $\mathbf{Z}^k$-invariant convex subspace $E \subset X$, isometric to k-dimensional Euclidean space, such that $\mathbf{Z}^k$ acts on it as a cocompact lattice of translations. □

We now discuss the more difficult case that indices of type 1 occur. As explained above, we can assume that no indices of type 0 occur and that $\Lambda_i^* \cong \mathbf{R}$ for all indices of type 2.

Choose a vertex $x_0 \in X$ as an origin. For indices of type 2 choose the parameter on the above geodesics σ_i such that $\sigma_i(0) = r_i(x_0)$. For indices of type 1 we denote by $\omega_i \in \Lambda_i^*(\infty)$ the corresponding fixed point. For these indices, we let $\sigma_i \colon [0, \infty) \to \Lambda_i^*$ be a unit speed geodesic ray with $\sigma_i(0) = r_i(x_0)$ and $\sigma_i(\infty) = \omega_i$.

We set $F = \operatorname{im} \sigma_1 \times \cdots \times \operatorname{im} \sigma_n$. Note that F is a closed and convex subspace of $\prod \Lambda_i^*$. We also define a geodesic ray

$$\sigma \colon [0, \infty) \to F \quad \text{by} \quad \sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t)).$$

By construction, $\sigma(0) = r(x_0)$.

LEMMA 4.4. $\operatorname{Stab}(\sigma_i(t)) \to \Delta_i$ *and* $\operatorname{Stab}(\sigma(t)) \to \Delta$ *as* $t \to \infty$, *where the limit of groups is understood as the union of increasing family.*

Proof. Let $\phi \in \Delta_i$. Then ϕ fixes $\omega_i = \sigma_i(\infty)$. Therefore $\phi \circ \sigma_i$ is asymptotic to σ_i. Now Λ_i^* is a tree, hence $\phi \circ \sigma_i(t) = \sigma_i(t + c)$ for all t sufficiently large, where c is some constant independent of t. Since $\phi \in \Delta_i$, $c = 0$ and therefore $\phi \in \operatorname{Stab}(\sigma_i(t))$ for all t sufficiently large. □

COROLLARY 4.5. *There exists a sequence* (x_m) *in* X *such that* $\operatorname{Stab}(x_m) \to \Delta$.

Proof. We observe that $\operatorname{Stab}(x) \subset \Delta$ for all $x \in X$. Now the assertion follows immediately from Proposition 3.5 and Lemma 4.4. □

LEMMA 4.6. *If the group* Γ *fixes precisely one point* $\omega_i \in \Lambda_i^*(\infty)$, *then* $\Delta \cap \operatorname{Stab}(\sigma_i(t))$ *has infinitely many jumps as* $t \to \infty$.

Proof. Let $\phi \in \Delta \subset \Delta_i$. By Lemma 4.4 there is $t_\phi \geq 0$ such that $\phi \in \operatorname{Stab}(\sigma_i(t))$ for all $t \geq t_\phi$. Hence if $\Delta \cap \operatorname{Stab}(\sigma_i(t)) = \Delta \cap \operatorname{Stab}(\sigma_i(t'))$ for all t, t' sufficiently large, then $\Delta \subset \operatorname{Stab}(\sigma_i(t))$ for all t sufficiently large. By Lemma 4.2, Γ either fixes a point of Λ_i^*, which is excluded by our reductions above, or Γ fixes exactly two points of $\Lambda_i^*(\infty)$, which is in contradiction to the assumption. □

LEMMA 4.7. *Let* (x_m) *be a sequence in* X *such that* $\operatorname{Stab}(x_m) \to \Delta$ *and* $\gamma_m \colon [0, s_m] \to X$ *be the unit speed geodesic from* x_0 *to* x_m, *where* $s_m = d(x_0, x_m)$. *Then given a constant* $t_0 > 0$, *there exists* m_0 *such that* $s_m \geq t_0$ *and* $r \circ \gamma_m([0, t_0]) \in F$ *for all* $m \geq m_0$.

Proof. For those i for which Γ fixes exactly one point $\omega_i \in \Lambda_i^*(\infty)$ we choose $\phi_i \in \Delta$ such that $\phi_i \notin \mathrm{Stab}(\sigma_i(t))$ for $t \le t_0$, see Lemma 4.6. By assumption, there is m_0 such that $\phi_i \in \mathrm{Stab}(x_m)$ for all $m \ge m_0$ and all such i. Now $r_i \circ \gamma_m$ is a monotonic curve in Λ_i^* from $\sigma_i(0) = r_i(x_0)$ to $r_i(x_m)$. By equivariance of r_i, $\phi_i \in \mathrm{Stab}(r_i(x_m))$ for all $m \ge m_0$. On the other hand, $r_i \circ \sigma$ has speed ≤ 1, hence by the choice of t_0, $s_m \ge t_0$ and $r_i(\gamma_m(t)) \in \sigma_i([0, t_0])$ for $0 \le t \le t_0$.

The claim follows since the image of r_i is σ_i for those i for which Γ fixes exactly two ends of Λ_i^*. □

LEMMA 4.8. *Given $\phi \in \Gamma$, there is a constant $c = c_\phi$ such that $d(\phi(p), p) \le c$ for all $p \in F$.*

Proof. We show that $d_i(\phi(p), p) \le c_i$ for each point p in the image of σ_i. This is clear for those indices i for which Γ fixes exactly two ends of Λ_i^*. Consider some other index i. Then σ_i is defined on $[0, \infty)$.

If ϕ is elliptic on Λ_i^*, then $\phi \in \Delta_i$. By Lemma 4.4, there exists a constant t_ϕ such that ϕ fixes $\sigma_i(t)$ for all $t \ge t_\phi$. We conclude that $d_i(\phi(p), p) \le 2t_\phi$ for each point p in the image of σ_i.

We assume now that ϕ is axial on Λ_i^* and let ρ be an axis of ϕ in Λ_i^*. We parametrize ρ such that $\rho(\infty) = \omega_i$. Since Λ_i^* is a tree and $\sigma_i(\infty) = \rho(\infty)$, we can actually choose the parameter such that $\sigma_i(t) = \rho(t)$ for all $t \ge t_\phi$, where t_ϕ is an appropriate constant. Now $\phi(\rho(t)) = \rho(t+\tau)$ for some constant τ independent of t. We conclude that $d_i(\phi(p), p) \le 2t_\phi + \tau$ for each point p in the image of σ_i. □

PROPOSITION 4.9. *Suppose that indices of type* 1 *occur. Then*

(1) *Δ does not fix a point of X;*

(2) *Γ fixes a point in $X(\infty)$. More precisely, if (x_m) is a sequence in X such that* $\mathrm{Stab}(x_m) \to \Delta$, *then after passing to a subsequence if necessary, (x_m) converges to a fixed point $\xi \in X(\infty)$ of Γ.*

Proof. The first assertion is an immediate consequence of Lemma 4.7. As for the proof of the second assertion, let (x_m) be a sequence in X with $\mathrm{Stab}(x_m) \to \Delta$. Let $\gamma_m \colon [0, s_m] \to X$ be the unit speed geodesic from x_0 to x_m as in Lemma 4.7. Note that $r \circ \gamma_m$ is a sequence of unit speed curves (with respect to the metric $d_{(2)}$, for which r restricted to any chamber of X is an isometry) in $\prod \Lambda_i^*$. For each constant $t_0 > 0$, $r \circ \gamma_m([0, t_0])$ is contained in F for all m sufficiently large. Now F is locally compact, hence a subsequence of the sequence of curves $r \circ \gamma_m$ converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics γ_m converges locally uniformly. By definition, this means that the corresponding subsequence of (x_m) converges to a point $\xi \in X(\infty)$.

Let $\phi \in \Gamma$ and choose $c = c_\phi$ as in Lemma 4.8. Let $t_0 > 0$ be given. By Lemma 4.8 we have $r \circ \gamma_m(t_0) \in F$ for all $m \geq m_0$. By Proposition 3.4 and Lemma 4.8, we have $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{n} c_\phi$ for all such m. Now c_ϕ is independent of t_0, hence $\phi(\xi) = \xi$. □

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1, $\Delta \cong \ker h$ consists precisely of the elliptic elements of Γ. If indices of type 1 do not occur, then Proposition 4.3 applies: If $k = 0$, then $\Gamma \cong \Delta$ fixes a point of X and possibility (1) holds. If $k > 0$, then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that $\mathrm{Stab}(x) \neq \Delta$ for any $x \in X$ in this case since Δ would have a fixed point otherwise.

5. Parallel transport in a cubical manifold and the proof of Theorem 3

Let X be a cubical manifold of dimension n. Given two chambers P and Q in X with a common face of dimension $n-1$, we define $t_{PQ}\colon P \to Q$ to be the *translation* which moves each point p of P along the unit geodesic segment starting at p and orthogonal to the common $(n-1)$-face of P to the end point in Q. The map t_{PQ} is an isomorphism and isometry of P with Q. Given a gallery $\pi = (P_1, \dots, P_n)$ in X, the *parallel transport* along π is the isomorphism $t_\pi\colon P_1 \to P_n$ given by

$$t_\pi := t_{P_{n-1}P_n} \circ \cdots \circ t_{P_2P_3} \circ t_{P_1P_2} .$$

LEMMA 5.1. *Let X be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. Then for any two chambers P and Q in X, the parallel transport t_π along a gallery π connecting P and Q is independent of π.*

Proof. It is enough to show that the parallel transport along any closed gallery is the identity. Let π be such a gallery with initial and final chamber P.