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COUNTING PATHS IN GRAPHS

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ABSTRACT. We give a simple combinatorial proof of a formula that extends a result by Grigorchuk [Gri78a, Gri78b] relating cogrowth and spectral radius of random walks. Our main result is an explicit equation determining the number of 'bumps' on paths in a graph: in a d-regular (not necessarily transitive) non-oriented graph let the series $G(t)$ count all paths between two fixed points weighted by their length t^{length}, and $F(u,t)$ count the same paths, weighted as $u^{\text{number of bumps}} t^{\text{length}}$. Then one has

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}\,.$$

We then derive the circuit series of 'free products' and 'direct products' of graphs. We also obtain a generalized form of the Ihara-Selberg zeta function [Bas92, FZ98].

1. INTRODUCTION

Let $\Gamma = \mathbf{F}_S/N$ be a group generated by a finite set S, where $\mathbf{F}_S$ denotes the free group on S. Let f_n be the number of elements of the normal subgroup N of $\mathbf{F}_S$ whose minimal representation as words in $S \cup S^{-1}$ has length n; let g_n be the number of (not necessarily reduced) words of length n in $S \cup S^{-1}$ that evaluate to 1 in Γ; and let $d = |S \cup S^{-1}| = 2|S|$. The numbers

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{f_n}\,, \qquad \nu = \frac{1}{d}\limsup_{n\to\infty} \sqrt[n]{g_n}$$

are called the *cogrowth* and *spectral radius* of (Γ, S). The Grigorchuk Formula [Gri78b] states that

$$\nu = \begin{cases} \frac{\sqrt{d-1}}{d}\left(\frac{\alpha}{\sqrt{d-1}} + \frac{\sqrt{d-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{d-1}\,, \\ \frac{2\sqrt{d-1}}{d} & \text{else}\,. \end{cases} \tag{1.1}$$