

# 5. Graphs and matrices

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.07.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Similarly, letting  $\mathfrak{F}_{x,e,y}$  count the paths from  $x$  to  $y$  that start with the edge  $e$ ,

$$\begin{aligned}\mathfrak{F}_{x,y} &= \delta_{x,y} + \sum_{e \in E(\mathcal{X}): e^\alpha = x} \mathfrak{F}_{x,e,y}, \\ \mathfrak{F}_{x,e,y} &= e^\ell (\mathfrak{F}_{e^\omega,y} + (u-1)\mathfrak{F}_{e^\omega,\bar{e},y}), \\ \mathfrak{F}_{e^\omega,\bar{e},y} &= \bar{e}^\ell (\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y});\end{aligned}$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = (1 - (u-1)^2(e\bar{e})^\ell)^{-1} (e^\ell \mathfrak{F}_{e^\omega,y} + (u-1)(e\bar{e})^\ell \mathfrak{F}_{x,y}),$$

which we insert in the first line to obtain

$$K_x^{-1} \mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}): e^\alpha = x} \frac{e^\ell}{1 - (u-1)^2(e\bar{e})^\ell} K_{e^\omega} \cdot K_{e^\omega}^{-1} \mathfrak{F}_{e^\omega,y}.$$

Thus if we let

$$(4.1) \quad e^{\ell'} = \frac{e^\ell}{1 - (u-1)^2(e\bar{e})^\ell} K_{e^\omega}, \quad A' = \sum_{e \in E(\mathcal{X})} [e^{\ell'}]_{e^\alpha}^{e^\omega},$$

we obtain

$$(4.2) \quad (K_x^{-1} \mathfrak{F}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A'}$$

and the proof is finished in the case that  $\mathcal{X}$  is finite, because the matrix  $A'$  is precisely that obtained from  $A$  by substituting  $\ell'$  for  $\ell$ .

If  $\mathcal{X}$  has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by  $\mathfrak{F}_{\star,\dagger}^n(\ell)$  and  $\mathfrak{G}_{\star,\dagger}^n(\ell')$  the enriched path series and path series respectively in  $\mathcal{B}(\star, n)$ , and write

$$K_\star \cdot \mathfrak{F}(\ell) = \lim_{n \rightarrow \infty} \mathfrak{F}_{\star,\dagger}^n(\ell) = \lim_{n \rightarrow \infty} \mathfrak{G}_{\star,\dagger}^n(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

## 5. GRAPHS AND MATRICES

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand ' $X/Y$ ' as ' $X \cdot Y^{-1}$ '.

DEFINITION 5.1. Let  $\mathcal{X}$  be a finite graph. The *edge-adjacency* and *inversion* matrices of  $\mathcal{X}$ , respectively  $B$  and  $J$ , are  $E(\mathcal{X}) \times E(\mathcal{X})$  matrices over  $\mathbf{Z}$  defined by

$$B_{e,f} = \begin{cases} 1 & \text{if } e^\omega = f^\alpha \\ 0 & \text{else,} \end{cases} \quad J_{e,f} = \begin{cases} 1 & \text{if } \bar{e} = f \\ 0 & \text{else.} \end{cases}$$

The *vertex-adjacency* and *degree* matrices of  $\mathcal{X}$ , respectively  $A$  and  $D$ , are  $V(\mathcal{X}) \times V(\mathcal{X})$  matrices over  $\mathbf{Z}$  defined by

$$A_{v,w} = |\{e \in E(\mathcal{X}) \mid e^\alpha = v \text{ and } e^\omega = w\}|, \quad D_{v,w} = \begin{cases} \deg(v) & \text{if } v = w, \\ 0 & \text{else.} \end{cases}$$

A *cycle* is the equivalence class of a circuit under cyclic permutation of its edges. A *proper cycle* is a cycle all of whose representatives are proper circuits. A cycle is *primitive* if none of its representatives can be written as  $\pi^k$  for some  $k \geq 2$ . The *cyclic bump count*  $\text{cbc}(\pi)$  of a circuit  $\pi = (\pi_1, \dots, \pi_n)$  is

$$\text{cbc}(\pi) = |\{i = 1, \dots, n \mid \pi_i = \overline{\pi_{i+1}}\}|,$$

where the edge  $\pi_{n+1}$  is understood to be  $\pi_1$ .

The matrices given above are related to paths in  $\mathcal{X}$  as follows: Consider first the matrix

$$M = \mathbf{1} - (B - (1 - u)J)t.$$

Then the  $(e, f)$  coefficient of  $M^{-1}$  is precisely

$$\sum_{\pi: \pi_1=e, \pi^\omega=f^\alpha} u^{\text{bc}(\pi f)} t^{|\pi|}.$$

This is clear because the series expansion of  $M^{-1}$  is the sum of sequences of  $(B - J)t$  (contributing edges with no bump) and  $Jut$  (contributing edges with bumps), with an extra factor of  $u$  in case the path ends in  $\bar{f}$ . As a consequence,

LEMMA 5.2. *Let*

$$X_E = \frac{\mathbf{1} + (1 - u)Jt - M}{Mt} = \frac{B}{\mathbf{1} - (B - (1 - u)J)t}.$$

*Then the  $(e, f)$  coefficient of  $X_E$  counts the non-trivial paths starting with  $e$  and ending at  $f^\alpha$ , with  $t$ -weight shifted one down:*

$$(X_E)_{e,f} = \sum_{\pi: \pi_1=e, \pi^\omega=f^\alpha} u^{\text{bc}(\pi)} t^{|\pi|-1}.$$

Likewise, consider the matrix

$$P = \mathbf{1} - At + (1 - u)(D - (1 - u)\mathbf{1})t^2 .$$

The following lemma will be a consequence of the computations in the next section.

LEMMA 5.3. *Let*

$$X_V = \frac{(1 - (1 - u)^2 t^2)\mathbf{1} - P}{Pt} = \frac{A - (1 - u)Dt}{\mathbf{1} - At + (1 - u)(D - (1 - u)\mathbf{1})t^2} .$$

*Then the  $(v, w)$  coefficient of  $X_V$  counts the non-trivial paths starting at  $v$  and ending at  $w$ , with  $t$ -weight shifted one down:*

$$(X_V)_{v,w} = \sum_{\pi: \pi^\alpha = v, \pi^\omega = w} u^{\text{bc}(\pi)} t^{|\pi| - 1} .$$

*Proof.* We will show the matrix  $\mathbf{1} + X_V t$  has as  $(v, w)$  coefficient the enriched path series from  $v$  to  $w$ . By simple calculation

$$\mathbf{1} + X_V t = \frac{\mathbf{1} - (1 - u)^2 t^2}{\mathbf{1} - At + (1 - u)(D - (1 - u)\mathbf{1})t^2} = \frac{K^{-1}}{\mathbf{1} - A'} ,$$

where  $K$  and  $A'$  are given by

$$K = \frac{\mathbf{1} + (1 - u)(D - 1 + u)t^2}{1 - (1 - u)^2 t^2} , \quad A' = \frac{AKt}{1 - (1 - u)^2 t^2} .$$

$K$  is a diagonal matrix and the coefficient  $K_{x,x}$  is precisely  $K_x$  for the length labelling, while the matrix  $A'$  is the matrix of (4.1) in the previous section. The result then follows from Equation (4.2).  $\square$

In particular, the two matrices  $X_E$  and  $X_V$  have the same trace, as this trace counts all the non-trivial circuits  $\pi$  in  $\mathcal{X}$ , with weight  $u^{\text{bc}(\pi)} t^{|\pi| - 1}$ .

We now state and prove an extension of a theorem by Bass [Bas92, FZ98, Nor96]:

THEOREM 5.4. *Let  $\mathcal{C}$  be a set of representatives of primitive cycles in  $\mathcal{X}$ , and form the zeta function of  $\mathcal{X}$*

$$\zeta(u, t) = \prod_{\gamma \in \mathcal{C}} \frac{1}{1 - u^{\text{cbc}(\gamma)} t^{|\gamma|}} .$$

(The choice of representatives does not change the zeta function.) Then  $\zeta^{-1}$  is a polynomial in  $u$  and  $t$  and can be expressed as

$$(5.1) \quad \frac{1}{\zeta(u, t)} = \det M$$

$$(5.2) \quad = (1 + (1 - u)t)^n (1 - (1 - u)^2 t^2)^{m - |V(\mathcal{X})|} \det P,$$

where

$$n = |\{e \in E(\mathcal{X}) \mid e = \bar{e}\}|, \quad 2m = |\{e \in E(\mathcal{X}) \mid e \neq \bar{e}\}|.$$

The special case  $u = n = 0$  of this result was stated and proved in the given sources. We will prove the general statement, using a result of Shimson Amitsur:

**THEOREM 5.5** (Amitsur [Ami80,RS87]). *Let  $X_1, \dots, X_k$  be square matrices of the same dimension over an arbitrary ring. Let  $S$  contain one representative up to cyclic permutation of words over the alphabet  $\{1, \dots, k\}$  that are primitive, i.e. such that none of their cyclic permutations are proper powers of a word ( $S$  is infinite as soon as  $k > 1$ ). For  $p = i_1 \dots i_n \in S$  set  $X_p = X_{i_1} \dots X_{i_n}$ . Then*

$$\det(\mathbf{1} - (X_1 + \dots + X_k)t) = \prod_{p \in S} \det(\mathbf{1} - X_p t^{|p|}),$$

considered as an equality of power series in  $t$  over the matrix ring.

The equality (5.1) then follows; indeed, for all edges  $e \in E(\mathcal{X})$  let  $X_e$  be the  $E(\mathcal{X}) \times E(\mathcal{X})$  matrix whose  $e$ -th row is the  $e$ -th row of  $B - (1 - u)J$ , and whose other rows are 0. Then clearly  $\mathbf{1} - \sum_{e \in E(\mathcal{X})} X_e t = M$  and, for any sequence of edges  $\pi$ ,

$$\det(\mathbf{1} - X_\pi t^{|\pi|}) = \begin{cases} 1 - u^{\text{cbc}(\pi)} t^{|\pi|} & \text{if } \pi \text{ is a circuit,} \\ 1 & \text{else,} \end{cases}$$

so equality of  $\zeta(u, t)$  and  $\det M$  follows from Amitsur's Theorem.

To prove (5.2), we use the following result, whose proof relies on Newton's formulas relating the trace of powers of  $X$  and the characteristic polynomial of  $X$ :

PROPOSITION 5.6 ([Ami80, Equation 4.4]). *Let  $X$  be a power series in  $t$  over a matrix ring, such that  $X(0) = \mathbf{1}$ . Then*

$$\det X = \exp\left(-\int \operatorname{tr}\left(\frac{\mathbf{1} - X}{Xt}\right) dt\right),$$

where the integration is the formal linear operation on power series that maps  $t^r$  to  $t^{r+1}/(r+1)$ .

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned} \frac{\det M}{(1 + (1 - u)t)^n (1 - (1 - u)^2 t^2)^m} &= \det \frac{M}{\mathbf{1} + (1 - u)Jt} \\ &= \exp\left(-\int \operatorname{tr} \frac{\mathbf{1} + (1 - u)Jt - M}{Mt} dt\right) \\ &= \exp\left(-\int \text{series counting non-trivial circuits,} \right. \\ &\quad \left. \text{length shifted down by one} dt\right) \\ &= \exp\left(-\int \operatorname{tr} \frac{(1 - (1 - u)^2 t^2)\mathbf{1} - P}{Pt} dt\right) \\ &= \det \frac{P}{1 - (1 - u)^2 t^2} = \frac{\det P}{(1 - (1 - u)^2 t^2)^{|V(X)|}}. \end{aligned}$$

## 6. SECOND PROOF OF THEOREM 2.4

Let  $P = [\star, \dagger]$  be the set of paths in  $\mathcal{X}$  from  $\star$  to  $\dagger$ . As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in  $\Pi = \mathbf{Z}[[P]]$ , the  $\mathbf{Z}$ -module of functions from the set of paths to  $\mathbf{Z}$ . We embed subsets of  $P$  in  $\Pi$  by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A, \quad \text{with } (\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\mathcal{B}$  be the subset of bounded non-negative elements of  $\Pi$  (i.e. functions  $f$  such that there is a constant  $N$  with  $0 \leq (\pi)f < N$  for all paths  $\pi$ ). If  $\ell$  is a complete labelling of  $\mathcal{X}$ , there is an induced labelling  $\ell_*: \mathcal{B} \rightarrow \mathbf{k}$  given by

$$(f)\ell_* = \sum_{\pi \in P} (\pi)f \pi^\ell.$$

Note that the sum, although infinite, defines an element of  $\mathbf{k}$  due to the fact that  $\ell$  is complete.