12. ACKNOWLEDGEMENTS

the amalgamated product $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ is isomorphic to $\mathbf{Z}^2$. The circuit series of $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ have been calculated explicitly and are algebraic. The circuit series of $\mathcal{X}$ was shown in Section 10 to be transcendental; so there can exist no algebraic definition of $G_{\mathcal{X}}$ in terms of $G_{\mathcal{D}}$, $G_{\mathcal{E}}$ and $G_{\mathcal{F}}$. However, there exists some relations between these series, as given by [Voi90, Theorem 5.5].

Given a graph $\mathcal{X}$, one can construct a graph $\mathcal{X}^{(k)}$ on the same vertex set, and with edge set the set of paths of length $\leq k$ in $\mathcal{X}$. Is there some simple relation between the path series of $\mathcal{X}$ and of $\mathcal{X}^{(k)}$? This could be useful for example to obtain asymptotics about the cogrowth of a group subject to enlargement of generating set [Cha93].

The equation (9.2) corresponds to Voiculescu's R-transform [Voi90]. His S-transform, in terms of graphs, corresponds to $\mathcal{E} * \mathcal{F}$ with as edge set all sequences (e,f) and (f,e), for $e \in E(\mathcal{E})$ and $f \in E(\mathcal{F})$. Is there an analogue to Theorem 9.2 in this context?

Finally, (9.2) computes the circuit series of a free product in terms of the circuit series of the factors. A more complicated formula yields the path series of a free product in terms of the path series of the factors. Such considerations give another derivation of the results in Section 8.

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Added in proof. Recently Vaughan Jones has obtained very similar results in the context of planar algebras, for which some 'path' and 'proper path' series give the Hilbert-Poincaré series of a planar algebra over different subalgebras (see *Planar Algebras I*; preprint at http://www.math.berkeley.edu/~vfr/plnalg1.ps).