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TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

by Paul SCHMUTZ SCHALLER

1. INTRODUCTION

There are a number of ways to define the Teichmüller space of Riemann surfaces. In this paper I treat an approach which is less common than others. Let Γ be a Fuchsian group which uniformizes a closed Riemann surface of genus g. Then a fundamental domain for Γ is chosen in a canonical way, namely as a polygon with $4g$ sides such that opposite sides are identified. The Teichmüller space T_g of closed Riemann surfaces of genus g is then constructed by varying these polygons.

This construction of T_g by polygons was first done by Coldewey and Zieschang in an annex in [17], see also [18]; the construction includes the proof that T_g is homeomorphic to $\mathbf{R}^{6g-6}$. In [2], Buser gave a different, however indirect proof. Here, I propose a new construction and a new proof which is, in my eyes, easier and more transparent than the original one of Coldewey and Zieschang.

The main idea is the following. Let $P(g)$ be a canonical polygon of $4g$ sides which is the fundamental domain of a Fuchsian group uniformizing a closed Riemann surfaces of genus g (the definition of $P(g)$ will include some technical subtleties, to be discussed in Section 3). Then "triangulate" $P(g)$ into $4g-4$ triangles and one quadrilateral S. This can be done in such a way that these triangles are determined by $6g-5$ positive real numbers (corresponding to the lengths of the sides of the triangles) with the condition that the different triangle inequalities hold. It turns out that these $6g-5$ lengths, *taken as homogeneous parameters*, provide a parametrization of the Teichmüller space T_g. Since the set of reals for which the different triangle

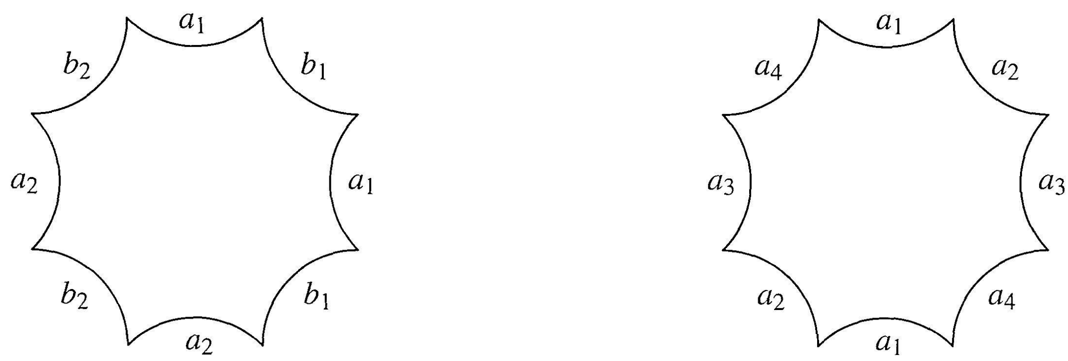

FIGURE 1

On the left hand side: usual identification

On the right hand side: identification chosen in this paper

inequalities hold is open and convex, this also proves that T_g is homeomorphic to $\mathbf{R}^{6g-6}$.

Let P be a polygon of $4g$ sides which is the fundamental domain for a Fuchsian group Γ uniformizing a closed Riemann surface M of genus g. This means that we can write

$$M = \mathbf{H}/\Gamma$$

where $\mathbf{H}$ is the upper halfplane. Usually, P is chosen such that the identification of the sides of P is that of the polygon on the left hand side in Figure 1. The construction described above would equally work for these polygons. For the following reasons I prefer to choose the identification (compare the polygon on the right hand side of Figure 1) such that opposite sides are identified. First the sides of P correspond to simple (this means with no selfintersections) closed curves in M and if opposite sides are identified, then these simple closed curves intersect transversally (which is not the case with the usual identification). Secondly, the vertices of P correspond to a (unique) point Q in M; with the usual identification, Q is completely arbitrary while with the identification chosen here, there is a natural choice for Q in the case of hyperelliptic Riemann surfaces, namely, as a Weierstrass point. See Section 6 for details.

In this paper, I only treat the case of Fuchsian groups which uniformize closed Riemann surfaces. In a straightforward way, the construction and proof could be extended to all finitely generated Fuchsian groups. Note that concerning the original construction and proof in [17] (mentioned above) the corresponding generalization has been worked out by Coldewey in his thesis [3].

The paper is structured as follows. In Section 2 the basic definitions of hyperbolic geometry and Fuchsian groups are given. Section 3 defines the canonical polygons. Section 4 provides the necessary material from hyperbolic trigonometry, it contains also some lemmas needed later. Section 5 contains the proof of the main theorem and Section 6 gives some applications, mainly concerning hyperelliptic Riemann surfaces. More precisely, I give a new proof of a geometric characterization of hyperelliptic Riemann surfaces which first appeared in [14] (I thank very much Feng Luo who, by his comments on [14], has contributed to the idea of this new proof). I also show (and this is a new result) that the Teichmüller space T_g for $g = 2$ can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. This is the optimum parametrization of Teichmüller space by geodesic length functions which one can expect.

I spoke about the content of this paper in lectures of the Troisième Cycle Romand de Mathématiques (Lausanne 1997); I thank the participants for their comments.

2. Hyperbolic geometry and Fuchsian groups

The material of this section and of parts of the following section is standard, see for example [1], [4], [5], [6], [7], [8], [15].

Definition. (i) $\mathbf{H} = \{z = (x, y) \in \mathbf{C} : y > 0\}$ denotes the *upper halfplane.* The *hyperbolic metric* on $\mathbf{H}$ is given by

$$dz = \frac{1}{y}(dz)_E$$

where $(dz)_E$ is the standard Euclidean metric on $\mathbf{C}$ and y is the imaginary part of z.

(ii) Define

$$\mathrm{SL}(2,\mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1;\ a, b, c, d \in \mathbf{R} \right\}$$

and

$$\mathrm{PSL}(2,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})/\!\sim$$

with $A \sim B$ if and only if $A = \pm B$ for $A, B \in \mathrm{SL}(2,\mathbf{R})$. Let $\gamma \in \mathrm{SL}(2,\mathbf{R})$,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$