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Addition in $K^*(X, G)$ is given by disjoint union of K -cocycles. Further,

$$K^*(X, G) = K^0(X, G) \oplus K^1(X, G),$$

where $K^i(X, G)$ is the subgroup of $K^*(X, G)$ determined by all K -cocycles (Z, ξ, f) with $\xi \in V_G^i(T^*Z \oplus f^*T^*X)$. The natural homomorphism of abelian groups

$$K^i(X, G) \rightarrow K_i[C_0(X) \rtimes G]$$

is defined by

$$(Z, \xi, f) \mapsto \mu(Z, \xi, f).$$

CONJECTURE. For any G -manifold X , $\mu: K^i(X, G) \rightarrow K_i[C_0(X) \rtimes G]$ is an isomorphism.

This conjecture is known to be true if X is a proper G -manifold. If X is proper there is a commutative diagram

$$\begin{array}{ccc} K^*(X, G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ & i_t \searrow & \swarrow \alpha \\ & K_G^*(X) & \end{array}$$

in which each arrow is an isomorphism. $i_t: K^*(X, G) \rightarrow K_G^*(X)$ maps a K -cocycle (Z, ξ, f) to its topological index, and $\alpha \circ \mu: K^*(X, G) \rightarrow K_G^*(X)$ maps a K -cocycle (Z, ξ, f) to its analytic index. If G is compact then any G -manifold is proper and commutativity of the diagram is equivalent to the Atiyah-Singer index theorems of [6], [7], [8].

3. HOMOTOPY QUOTIENT

Let W be a topological space. $V^0(W)$ denotes the collection of all complex vector bundles (E_0, E_1, σ) on W with compact support. Thus E_0, E_1 are complex vector bundles on W and $\sigma: E_0 \rightarrow E_1$ is a morphism of complex vector bundles with $\text{Support}(\sigma)$ compact, where

$$\text{Support}(\sigma) = \{p \in W \mid \sigma: E_{0p} \rightarrow E_{1p} \text{ is not an isomorphism}\}.$$

Also $V^1(W) = V^0(W \times \mathbf{R})$.

Suppose given an \mathbf{R} -vector bundle F on W . Following [9], a *twisted* by F K -cycle on W is a triple (M, ξ, ϕ) such that:

- (1) M is a C^∞ -manifold without boundary;
- (2) $\phi: M \rightarrow W$ is a continuous map from M to W ;
- (3) $\xi \in V^*(T^*M \oplus \phi^*F)$.

As in [9] an equivalence relation is imposed on these twisted by F K -cycles to obtain the twisted by F K -homology of W :

$$K_*^F(W) = K_0^F(W) \oplus K_1^F(W).$$

$K_1^F(W)$ is the subgroup determined by all (M, ξ, ϕ) with $\xi \in V^i(T^*M \oplus \phi^*F)$. If F has a Spin^c -structure then $K_*^F(W)$ is isomorphic to $K_*(W)$, the K -homology of W .

With G as in §2 above, let EG be a contractible space on which G acts freely

$$EG \times G \rightarrow EG.$$

Given a G -manifold X , let G act on $EG \times X$ by

$$(p, x)g = (pg, xg)$$

($p \in EG, x \in X, g \in G$). The quotient space $[EG \times X]/G$ will be referred to as the homotopy quotient. Since T^*X is a G -vector bundle on X , the quotient $[EG \times T^*X]/G$ is a vector bundle on $[EG \times X]/G$. Denote this vector bundle by τ and consider the twisted by τ K -homology $K_*^\tau([EG \times X]/G)$. There is a map

$$K_*^\tau([EG \times X]/G) \rightarrow K^*(X, G).$$

This map is not quite canonical. First an orientation must be chosen for the Lie algebra of G , so assume that such an orientation has been chosen.

Let (M, ξ, ϕ) be a twisted by τ K -cycle on $[EG \times X]/G$. Now $EG \times X$ is the total space of a principal G -bundle over $[EG \times X]/G$ and this principal bundle can be pulled back via ϕ to yield a principal bundle Z over M

$$\begin{array}{ccc} EG \times X & \xleftarrow{\tilde{\phi}} & Z \\ \downarrow & & \downarrow \rho \\ [EG \times X] & \xleftarrow[\phi]{} & M. \end{array}$$

Let $\pi: EG \times X \rightarrow X$ be the projection and set $f = \pi \circ \tilde{\phi}$,

$$f: Z \rightarrow X.$$

$\xi \in V^*(T^*M \oplus \phi^*\tau)$ lifts to give $\tilde{\xi} \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$. Denote the bundle along the fibres of $\rho: Z \rightarrow M$ by F . This is a trivial vector bundle since,

for each $z \in Z$, F_z is canonically isomorphic to the Lie algebra of G . Using the orientation of this Lie algebra, F has a G -invariant Spin^c -structure so that $\tilde{\xi} \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$ determines $\eta \in V_G^*(F \oplus \rho^*T^*M \oplus f^*T^*X)$. Now $F \oplus \rho^*T^*M = T^*Z$, so (Z, η, f) is a K -cocycle for (X, G) . The map

$$K_*^T([EG \times X]/G) \rightarrow K^*(X, G)$$

is:

$$(M, \xi, \phi) \mapsto (Z, \eta, f).$$

This map has a dimension-shift in it. Set $\epsilon = \dim(G)$. Then with addition of indices mod 2 this map takes $K_i^T([EG \times X]/G)$ to $K^{i+\epsilon}(X, G)$.

LEMMA 1. *If G is torsion free then $K_*^T([EG \times X]/G) \rightarrow K^*(X, G)$ is an isomorphism.*

Proof. Let (Z, ξ, f) be a K -cocycle for (X, G) . The action of G on Z is proper, so each isotropy group is compact. Since G is assumed to be torsion free this implies that the action of G on Z is free. Hence Z is a G -principal bundle over G/Z , and thus Z maps equivariantly to EG . Combining this with $f: Z \rightarrow X$ we obtain a commutative diagram

$$\begin{array}{ccc} EG \times X & \longleftarrow & Z \\ \downarrow & & \downarrow \rho \\ [EG \times X] & \longleftarrow & Z/G. \end{array}$$

Denote the map of Z/G to $[EG \times X]/G$ by ϕ . Then $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$ determines $\xi' \in V_G^*(\rho^*T^*(Z/G) \oplus f^*T^*X)$. Since the action of G on Z is free ξ' descends to give $\theta \in V^*(T^*(Z/G) \oplus \tau)$. Then

$$(Z, \xi, f) \rightarrow (Z/G, \theta, \phi)$$

maps $K^*(X, G)$ to $K_*^T([EG \times X]/G)$ and provides an inverse to the map $K_*^T([EG \times X]/G) \rightarrow K^*(X, G)$. \square

REMARK 2. If G is the trivial one-element group then the isomorphism of the lemma becomes

$$K_*^{T^*X}(X) \cong K^*(X).$$

If X is a Spin^c -manifold then $K_*^{T^*X}(X) \cong K_*(X)$, so that in this case the isomorphism of the lemma becomes the Poincaré duality isomorphism $K_*(X) \cong K^*(X)$.

When G has torsion, the map $K_*^{\tau}([EG \times X]/G) \rightarrow K^*(X, G)$ can fail to be an isomorphism. The simplest example of this is obtained by taking X to be a point and $G = \mathbf{Z}/2\mathbf{Z}$.

When G has torsion, $K_*^{\tau}([EG \times X]/G)$ appears to be only a first approximation to $K^*(X, G)$ and $K_*[C_0(X) \rtimes G]$. The key point is that when G has torsion, there will be proper G -manifolds on which the G -action is not free.

4. SOLVABLE SIMPLY CONNECTED LIE GROUPS

The conjecture stated in §2 above is verified for (connected) solvable simply connected Lie groups by

PROPOSITION 1. *Let G be a (connected) solvable simply connected Lie group, and let X be a G -manifold. Then there is a commutative diagram*

$$\begin{array}{ccc} K^*(X, G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ \downarrow & & \downarrow \\ K^*(X) & \longrightarrow & K_*[C_0(X)] \end{array}$$

in which each arrow is an isomorphism.

The proof depends on

LEMMA 2. *Let G be a (connected) solvable simply connected Lie group, and let Z be a proper G -manifold. Then there exists a G -map from Z to G .*

Proof of Lemma 2. Since the action of G on Z is proper all isotropy groups are compact. G has no non-trivial compact subgroups, so the action of G on Z is free. Therefore Z is a principal G -bundle with base Z/G . As G is itself a contractible space on which G acts freely, there is a G -map from Z to G . \square

Proof of Proposition 1. In the diagram of the proposition the right vertical arrow is the Thom isomorphism of [13]. The lower horizontal arrow is the standard isomorphism which is valid for any locally compact Hausdorff topological space.