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DEFINITION 6. $K^*(V, F) = \Gamma(V, F)/\sim$. Addition in $K^*(V, F)$ is by disjoint union of K -cocycles. The natural homomorphism of abelian groups

$$K^i(V, F) \rightarrow K_i C^*(V, F)$$

is defined by

$$(Z, \xi) \rightarrow \mu(Z, \xi).$$

CONJECTURE. $\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$ is an isomorphism.

REMARK 7. Calculations of M. Pennington [25] and A. M. Torpe [32] verify the conjecture for certain foliations.

Given (V, F) , let BG be the classifying space of the holonomy groupoid G . Since ν is a G -vector bundle on V , ν induces a vector bundle τ on BG . As in §3 above there is then a natural map

$$K_*^\tau(BG) \rightarrow K^*(V, F).$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \rightarrow K^*(V, F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \rightarrow K^*(V, F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \rightarrow K^*(V, F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^* -algebra. If G is a Lie group and X is a G -manifold, then using A as coefficients there is both a geometric and an analytic K -theory for (X, G) . The analytic K -theory is the K -theory of the C^* -algebra $(C_0(X) \rtimes G) \otimes A$.

The geometric K -theory is obtained from K -cocycles (Z, ξ, f) where Z, f are as in §2 and $\xi = \{E_0 \xrightarrow{\sigma} E_1\}$ uses G -vector bundles E_0, E_1 on $T^*Z \oplus f^*T^*X$ such that the fibres of E_i are finitely generated projective modules over A . Denote this geometric K -theory by $K^*(X, G; A)$. The natural map

$$K^i(X, G; A) \rightarrow K_i[(C_0(X) \rtimes G) \otimes A]$$

is defined by using elliptic operators in the spirit of Mischenko-Fomenko [22]. We conjecture that this natural map is an isomorphism.

In the notation of Kasparov [18] the group denoted here by $K_*[C_0(X) \rtimes G]$ is $KK(\mathbb{C}, C_0(X) \rtimes G)$. For the K -homology group $KK(C_0(X) \rtimes G, \mathbb{C})$ there is a geometric group $K_*(X, G)$ which is the G -equivariant version of the topologically defined K -homology of [9]. Using transversally elliptic operators [2] one then obtains a natural map

$$K_*(X, G) \rightarrow KK(C_0(X) \rtimes G, \mathbb{C}).$$

We conjecture that this map is injective and that its image is dense (with respect to the natural topology) in $KK(C_0(X) \rtimes G, \mathbb{C})$.

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PROPOSITION 1 (A. Borel [10]). *Let G be a Lie group with $\pi_0 G$ finite and maximal compact subgroup H . If Z is any proper G -manifold then there exists a G -map from Z to $H \backslash G$.*

In §5 above this was proved for G a connected semi-simple Lie group with finite center. By the argument of §5, Borel's result implies:

COROLLARY 2. *Let G be a Lie group with $\pi_0 G$ finite. For any G -manifold X there is an isomorphism of abelian groups*

$$K_H^i(X \times (\mathfrak{h} \backslash \mathfrak{g})^*) \rightarrow K^i(X, G) \quad (i = 0, 1).$$

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