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REMARKS ON THE HAUSDORFF-YOUNG INEQUALITY

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§ 1. INTRODUCTION

A standard version of the Hausdorff-Young inequality for a locally compact commutative group G can be given as follows: for a fixed Haar measure in G, let $f \in L^1(G) \cap L^2(G)$; if $1 \leq p \leq 2$, $p' = p/(p-1)$, then

$$\|\widehat{f}\|_{p'} \leq \|f\|_p \tag{1}$$

where

$$\widehat{f}(\gamma) = \int_G f(x)\,\overline{\gamma(x)}\,dx\,, \qquad \gamma \in \widehat{G}\,, \tag{2}$$

$\widehat{G}$ being the dual group of G, endowed with a Haar measure which is such that for $p = p' = 2$, there is equality in (1); that this last condition can be met is one form of Plancherel's theorem in $L^2(G)$. Note that, for $1 \leq p \leq 2$, $\|f\|_p < \infty$ if f is in $L^1(G) \cap L^2(G)$, the latter space being dense in each $L^p(G)$, $1 \leq p \leq 2$. Hence, because of the Hausdorff-Young inequality (1), the Fourier transform $\mathcal{F}_p f$ can be defined uniquely for all $f \in L^p(G)$, $1 \leq p \leq 2$, in such a way that

$$\mathcal{F}_p \colon L^p(G) \to L^{p'}(\widehat{G}) \tag{3}$$

is a linear contraction with $\mathcal{F}_p f = \widehat{f}$ for all f in $L^1(G) \cap L^2(G)$. It is known that, for each $p \in [1,2], \mathcal{F}_p$ is injective and that if $f \in L^{p_1}(G) \cap L^{p_2}(G)$, $1 \leq p_1, p_2 \leq 2$, then $\mathcal{F}_{p_1} f = \mathcal{F}_{p_2} f$ a.e. on $\widehat{G}$; see [HR] vol. 2, chap. VIII ((31.26), p. 229; (31.31), p. 231). The purpose of the present note is to prove

(Thm. 1) by a very simple general argument that the operator $\mathcal{F}_p$ in (3) is surjective only in the following obvious cases: (i) $p = p' = 2$ or (ii) G finite. This fact is now well-known ([HR] vol. 2, p. 227, pp. 430–431); however, most of the known proofs of this depend on a careful analysis of the group G whereas our proof shows that this is an immediate consequence of a general theorem concerning the isomorphism of arbitrary L^p-spaces (stated in § 2). From this we deduce fairly simply that for any infinite locally compact commutative group G, the inequality (1) cannot be extended to the case $2 < p < \infty$; the exact statement is given as Thm. 2 in § 3. I have not seen this statement given in complete generality elsewhere, although it is highly likely to be known to many.

We set up the necessary notations in § 2, state and prove the facts alluded to above in § 3 and add a few historical comments in § 4; a short appendix (§ 5) is added to explain the L^p-isomorphism theorem stated in § 2.

We have not tried to extend our theorems to the case of G non-commutative, using for $\widehat{G}$ the set of all equivalence classes of continuous unitary irreducible representations of G. For G compact, this has been done (for our Thm. 1) in [HR] vol. 2, (37.19), p. 429; our analysis carries over to this case as well without any difficulty. However, we have preferred to leave out the non-commutative case entirely in this paper, except to make a few remarks on it in § 4.

§ 2. Notations and some known facts

Our reference for general functional analysis and integration theory is [DS] and that for group theory is [HR]. A measure space is a triple (X, Σ, μ) where Σ is a σ-algebra of subsets of the abstract set X and $\mu \colon \Sigma \to [0, \infty]$ is a σ-additive positive measure; no finiteness or σ-finiteness conditions will be imposed a priori on μ. Then $L^p(\mu)$, $1 \le p \le \infty$, will denote the usual Banach space associated with Σ-measurable complex-valued functions f defined on X with $\|f\|_p < \infty$, $\|f\|_p$ being the standard L^p-norm with respect to μ. If G is a locally compact commutative group (always supposed to be Hausdorff), $L^p(G)$, $1 \le p \le \infty$, will stand for the associated L^p-space obtained by fixing some Haar (invariant) measure on G, and $\widehat{G}$ will stand for the dual group, formed by the continuous homomorphisms (characters)

$$\gamma \colon G \to \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}.$$

For a given Haar measure on G, the Haar measure on $\widehat{G}$ will always be fixed in such a way that the Plancherel formula be valid in $L^2(G)$; if $f \in L^1(G)$, $\widehat{f}$ will be defined by (2) above.

Recall that for any measure space (X, Σ, μ), the dual Banach space of $L^p(\mu)$ is $L^{p'}(\mu)$ whenever $1 < p < \infty$, i.e. in symbols

$$(L^p(\mu))' = L^{p'}(\mu) \tag{4}$$

whether μ is σ-finite or not. Here and elsewhere,

$$p' = \frac{p}{p-1}, \qquad 1 < p < \infty,$$

and $1' = \infty$; for (4) to hold for $p = 1$, $p' = \infty$, one needs some conditions on μ (σ-finiteness of μ is sufficient but not necessary). Nevertheless,

$$(L^p(G))' = L^{p'}(G)$$

holds for all $p, 1 \leq p < \infty$, and any locally compact group G. We shall not, however, need this fact.

Two Banach spaces E, F are called isomorphic if there is an isomorphism $u \colon E \to F$ where u is a continuous linear bijection; it is well-known that $u^{-1} \colon F \to E$ is then automatically continuous. The following is proved in [C]: *if* (X, Σ, μ), $(Y, \mathcal{J}, \nu)$ *are any two measure spaces and* $1 \leq p, q \leq \infty$ *then* $L^p(\mu)$ *isomorphic to* $L^q(\nu)$ *implies necessarily that* $p = q$ *provided that* $L^p(\mu)$ *or* $L^q(\nu)$ *is infinite dimensional.*

We shall refer to this statement as the L^p*-isomorphism theorem*; as indicated in §5, this is an easy consequence of the theory of types and cotypes for Banach spaces. The same reasoning proves (cf. §5) that if (X, Σ, μ) is any measure space such that $L^1(\mu)$ is infinite dimensional and Y is any locally compact Hausdorff space then $L^1(\mu)$ cannot be isomorphic to $C_0(Y)$ (or to $C(Y)$). Here $C(Y)$ is the Banach space of all bounded complex-valued functions on Y endowed with the sup norm, and $C_0(Y)$ is the subspace of those continuous complex-valued functions in Y which vanish at ∞; if Y is compact, we put $C_0(Y) = C(Y)$.

Recall that if $f \in L^1(G)$, G any locally compact commutative group, then $\widehat{f} \in C_0(\widehat{G})$; cf. [HR] vol. 2, p. 212; this fact is sometimes referred to as the Riemann-Lebesgue lemma for G.

§3. MAIN THEOREMS

THEOREM 1. *Let G be a locally compact commutative group, $\widehat{G}$ its dual group, the Haar measures on $G, \widehat{G}$ being determined as in §2.*

(i) *If $1 \leq p < 2$, then the contraction operator $\mathcal{F}_p$ given by (3) is surjective if and only if G is a finite group.*

(ii) *$\mathcal{F}: L^1(G) \to C_0(\widehat{G})$ is surjective if and only if G is a finite group; here $\mathcal{F}f = \widehat{f}$.*

Proof. (i) It is known that $\mathcal{F}_p$ is always injective; cf. [HR] vol. 2, (31.31), p. 231. If $\mathcal{F}_p$ is surjective, then $\mathcal{F}_p$ will be an isomorphism between $L^p(G)$ and $L^{p'}(\widehat{G})$; since $p' \neq p$ if $1 \leq p < 2$, this implies, according to the L^p-isomorphism theorem of §2, that $L^p(G)$ and hence $L^{p'}(\widehat{G})$ are finite dimensional i.e. G (and hence $\widehat{G}$) are finite groups. On the other hand, if G is a finite commutative group then it is a well-known elementary fact (see [HR] vol. 1, p. 357) that $\widehat{G}$ is isomorphic to G so that, for any p, q in $[1, \infty]$, $L^p(G)$ and $L^q(\widehat{G})$ are then of the same finite dimension equal to the order of the group G; hence, in particular, if G is a finite commutative group, $L^p(G)$ is isomorphic to $L^{p'}(\widehat{G})$ for $1 \leq p < 2$; the isomorphism can be realized via $\mathcal{F}_p$ since $\mathcal{F}_p$ is injective and dim $L^p(G) =$ dim $L^{p'}(\widehat{G})$.

(ii) The proof here is perfectly similar; it uses the impossibility of an isomorphism between $L^1(\mu)$ and $C_0(Y)$ given in §2.

This completes the proof of Theorem 1.

The notations p', etc. are as in §2 for the following theorem as well; its proof uses the non-surjectivity given by Theorem 1 and an elementary inversion formula.

THEOREM 2. *Let G be an infinite commutative locally compact group and $2 < p < \infty$. Then no inequality of the form*

$$\|\widehat{f}\|_{p'} \leq M \|f\|_p \tag{5}$$

can hold for $f \in D$, D being a $L^p(G)$-dense linear subspace of $L^p(G) \cap L^1(G)$, whatever be the choice of M, $0 \leq M < \infty$.

Proof. We shall need the following simple facts:

(i) If $0 < a < c < b < \infty$ then, for any positive measure μ,

$$L^a(\mu) \cap L^b(\mu) \subset L^c(\mu).$$

This is evident from the following:

$$\int |f|^c d\mu = \int_{|f|\leq 1} |f|^c d\mu + \int_{|f|>1} |f|^c d\mu$$

$$\leq \int_{|f|\leq 1} |f|^a d\mu + \int_{|f|>1} |f|^b d\mu .$$

(ii) (Inversion formula for $L^2(G)$). If $f \in L^2(G)$ then

$$\tau\mathcal{F}_2(\mathcal{F}_2 f) = f$$

where $\tau g(x) = g(-x)$, $g\colon G \to \mathbf{C}$ being any function; cf. [HR] (31.17), p. 225.

(iii) If $\varphi \in L^a(G) \cap L^b(G)$, a, b being in $[1,2]$, then

$$\mathcal{F}_a\varphi = \mathcal{F}_b\varphi \qquad a.e.$$

This fact has already been explicitly mentioned in the introduction where an exact reference is given.

If (5) were to hold for $f \in D$, there would be a bounded linear operator T,

$$T\colon L^p(G) \to L^{p'}(\widehat{G})$$

such that

$$Tf = \widehat{f}, \qquad f \in D \subset L^p(G) \cap L^1(G).$$

Since $1 < p' < 2$, the Hausdorff-Young inequality gives a linear contraction S,

$$S\colon L^{p'}(\widehat{G}) \to L^p(G)$$

such that $S\varphi = \tau\mathcal{F}_{p'}\varphi$.

Now, if $f \in D$, f is in $L^2(G)$ (since $1 < 2 < p$; cf. (i) above) as well as in $L^1(G)$ (by hypothesis) so that

$$Tf = \widehat{f} \in L^2(\widehat{G}) \cap L^{p'}(\widehat{G}).$$

Thus, for $f \in D$,

$$S(Tf) = S(\mathcal{F}_2 f) = \tau\mathcal{F}_{p'}(\mathcal{F}_2 f) = \tau\mathcal{F}_2(\mathcal{F}_2 f) = f$$

by using the facts (ii) and (iii) given above. Since D is dense in $L^p(G)$ and the operator ST is continuous we deduce that

$$STf = f, \qquad f \in L^p(G),$$

which obviously implies that S must be surjective; this contradicts Theorem 1 thus establishing the impossibility of (5) for $f \in D$.

This completes the proof of Theorem 2.

REMARK. We observe that conversely, Theorem 1 can be deduced from Theorem 2; we shall not elaborate on this; however, our proof of Theorem 1 shows that its validity stems from a simple general result on L^p-spaces.

§4. HISTORICAL REMARKS

The inequality (1) was given first by Hausdorff [H] in 1923 for the groups $G = \mathbf{T}$ (with $\widehat{G} = \mathbf{Z}$) and $G = \mathbf{Z}$ (with $\widehat{G} = \mathbf{T}$). Hausdorff was inspired by the work of W.H. Young from 1912–13 who proved that the Fourier series of a function in L^p, $1 \le p \le 2$, had coefficients which were in $\ell^{p'}$ (and, in a suitable sense, vice-versa) for $p' = 2k$, a positive even integer, $p = 2k/(2k-1)$. Young did not formulate his results in terms of inequalities which were given first by Hausdorff (for all $p \in [1,2]$ and for the groups $G = \mathbf{T}$, $G = \mathbf{Z}$, i.e. for Fourier series). Hausdorff's proof, which is all but forgotten today, used Young's results for $p' = 2k$ and some of Young's techniques to carry out an interpolation argument for all the values of p, p', $1 \le p \le 2$, missing in Young's work. Hausdorff's paper [H] gives the exact references to W.H. Young's paper which were related to his work.

Shortly afterwards, after having heard of Hausdorff's inequalities, F. Riesz obtained independently (in [RF]) some Hausdorff-Young type inequalities, valid for series expansions in terms of arbitrary *bounded* orthogonal functions. This paper of F. Riesz was important not only because it showed that Hausdorff-Young type inequalities did not belong exclusively to the theory of Fourier series but also because F. Riesz (in collaboration with his colleague A. Haar) conjectured there the validity of a general "arithmetical" inequality for linear forms (in a finite number of variables) which they claimed to be enough for proving F. Riesz's theorem for orthogonal expansions.

It was this conjecture which seems to have led M. Riesz (F. Riesz's younger brother) to formulate and prove in 1927 ([RM]) his convexity theorem for bilinear forms and use it to deduce Hausdorff-Young-F. Riesz inequalities and many others. M. Riesz's work was exactly what A. Weil used in 1940 to establish (1) for general locally compact commutative groups in his book [W], p. 117. As is well-known, once the Plancherel theorem for a general $L^2(G)$, G locally compact commutative, is established (and this was done by Weil) the proof of (1) via M. Riesz's theorem is almost immediate. M. Riesz's work was simplified and much generalized by Thorin in 1938 (and later in 1948; exact references can be found in [DS] or in [HR]) which launched the later theory of interpolation of operators due to many well-known mathematicians which we shall not attempt to describe here. As regards Theorem 1 given in §3, it was proven by I. Segal in 1950 (for the part concerning $C_0(\widehat{G})$) and generally by E. Hewitt in 1954. The theorem was rediscovered by Rajagopalan in 1964; exact references to the papers of these authors can be found in [HR] vol. 2. The fact that the inequality (1) does not generalize to $p > 2$ had been foreseen in papers of 1918–19 by Carleman, Hardy and Littlewood, Landau for the case of $G = \mathbf{T}$ (exact references are in Hausdorff's paper [H]) where the work depends on the detailed study of the Fourier series of special continuous functions. However, I do not know of any explicit previous formulation and proof of Theorem 2 for arbitrary infinite locally compact commutative groups G; it is difficult to imagine that it has not been written down somewhere, since its proof is a straight-forward deduction from the inversion formula and the non-surjectivity theorem.

Theorem 1 has been generalized to the case of non-commutative compact groups G in [HR] vol. 2, (37.19), p. 429; now, $\widehat{G}$ is taken to be the set of all equivalence classes of continuous unitary irreducible representations of G. For G any locally compact unimodular group, Kunze (1958) has given the appropriate formulation of the Hausdorff-Young inequality (1) (see reference in [HR] vol. 2). If G is compact, $\widehat{G}$ as a set has the discrete topology and our proof of Theorem 1 carries over to this case. Theorem 2 in this case can be formulated as in [HR] vol. 2, (37.19) (iii), p. 429; its proof now is no more difficult than that of our Thm. 2. However, we do not intend to discuss the non-commutative case in any detail here.

We close this section by mentioning the remarkable later (1990) development around the Hausdorff-Young inequality due to Lieb [L]. Lieb has shown, generalizing considerably previous work of Babenko (1961) and Beckner (1975), that for the group $\mathbf{R}^n$ and for "Gaussian" transforms T more general than the Fourier transform, one has

$$\|Tf\|_{p'} \leq M_p \|f\|_p \tag{6}$$

where $M_p < 1$; Lieb has determined M_p exactly and has specified all the functions f for which equality obtains in (11). In particular, the L^p-Fourier transform in $\mathbf{R}^n$, $(1 \leq p < 2)$ turns out to be a strict contraction (a fact noticed by Babenko and Beckner) whose contraction coefficient (< 1) can be determined exactly; this is in sharp contrast to the situation in those locally compact abelian groups which have compact open subgroups where the corresponding Fourier transforms are just contractions. This has been studied in detail by Hewitt, Hirschmann, Ross ([HR] vol. 2, §43).

§5. Appendix

Here we outline a simple proof of the L^p-isomorphism theorem stated in §2; the proof uses the notion of type and cotype of Banach spaces and follows [C].

Definition. A Banach space E is of *type* p $(1 \le p \le 2)$ if there is a finite positive number C_p such that for all choices of $x_1, \dots, x_n$ in E, $n = 1, 2, \dots$, we have

$$2^{-n} \sum_{\varepsilon_1 \cdots \varepsilon_n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \le C_p \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{1/p},$$

where $\sum_{\varepsilon_1 \cdots \varepsilon_n}$ stands for the sum of the 2^n quantities obtained by letting each ε_j taking the values $+1$ or -1. E is said to have *exact type* p if it is of type p but not of type $\widetilde{p} > p$.

A Banach space E is of *cotype* q $(2 \le q \le \infty)$ if there is a finite positive number c_q such that for all choices of $x_1, \dots, x_n$ in E, $n = 1, 2, \dots$, we have

$$2^{-n} \sum_{\varepsilon_1 \cdots \varepsilon_n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \ge c_q \Big(\sum_{j=1}^{n} \|x_j\|^q \Big)^{1/q}.$$

E is said to have *exact cotype* q if it is of cotype q but not of cotype $\widetilde{q} < q$.

It is obvious that exact type or cotype is an isomorphism invariant. It can be shown that for any measure space (X, Σ, μ) giving rise to infinite dimensional $L^p(\mu)$-spaces we have the following:

- *$L^p(\mu)$ has exact type p if $1 \le p \le 2$, exact type 2 if $2 \le p < \infty$ and exact type 1 if $p = \infty$;*
- *$L^p(\mu)$ has exact cotype 2 if $1 \le p \le 2$, exact cotype p if $2 \le p < \infty$ and exact cotype ∞ if $p = \infty$.*

All this and more is completely proved in [C]; a reference for the general theory of types and cotypes is [DJT].

Suppose now that $L^p(\mu)$ and $L^q(\nu)$ are infinite dimensional and isomorphic where $1 \le p,\ q \le \infty$, $(X, \sum, \mu)$, $(Y, \mathcal{J}, \nu)$ being any two measure spaces; we shall prove that $p = q$. Without loss of generality, we may suppose that if $p \ne q$ then $p < q$; this would lead to a contradiction as shown below.

(i) If $1 \leq p < q \leq 2$ then

$$\text{exact type of } L^p(\mu) = p < \text{exact type of } L^q(\nu) = q,$$

which excludes any isomorphism between $L^p(\mu)$, $L^q(\nu)$.

(ii) If $1 \leq p < 2$, $2 \leq q < \infty$ then

$$\text{exact type of } L^p(\mu) = p < \text{exact type of } L^q(\nu) = 2,$$

which excludes any isomorphism between $L^p(\mu)$, $L^q(\nu)$.

(iii) If $2 \leq p < q < \infty$ then $1 < q' < p' \leq 2$; if $L^p(\mu)$, $L^q(\nu)$ were isomorphic then their duals $L^{p'}(\mu), L^{q'}(\nu)$ would be isomorphic, which is impossible in view of (i).

(iv) If $1 < p < \infty$, $q = \infty$ then $L^p(\mu)$ has exact type equal to $\min(p, 2) > 1$ whereas $L^\infty(\nu)$ has exact type 1; thus $L^p(\mu)$ is not isomorphic to $L^\infty(\nu)$ (a fact which is obvious on the grounds of reflexivity as well).

(v) Finally, let $p = 1$, $q = \infty$; then $L^1(\mu)$ is not isomorphic to $L^\infty(\nu)$ since the exact cotype of $L^1(\mu)$ is 2 and the exact cotype of $L^\infty(\nu)$ is ∞.

This completes the proof of the L^p-isomorphism theorem.

A proof that no infinite dimensional $L^1(\mu)$ can be isomorphic to any $C_0(Y)$ or $C(Y)$ (Y any locally compact Hausdorff space) can be based on the same ideas as (v) above. The exact cotype of $L^1(\mu)$ is 2 whereas the exact cotype of any infinite dimensional $C_0(Y)$ or $C(Y)$ is ∞ (exactly as in the case of $L^\infty(\mu)$). This excludes the possibility of any isomorphism between $L^1(\mu)$ and $C_0(Y)$ or $C(Y)$.

REMARK. The L^p-isomorphism theorem seems to be known to various specialists; however, I know of no explicit formulation or proof of it in complete generality except for that in [C].

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