

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 46 (2000)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE WITT GROUP OF LAURENT POLYNOMIALS  
**Autor:** Ojanguren, Manuel / Panin, Ivan  
**Kapitel:** 5. The Witt group of extended spaces  
**DOI:** <https://doi.org/10.5169/seals-64807>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 18.04.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

PROPOSITION 4.7. *The isomorphisms*

$$\partial_M: \operatorname{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \rightarrow \operatorname{Hom}_A(M, A)$$

*induce a surjective homomorphism*

$$\partial^W: W_{\text{tors}}(A[t]) \rightarrow W(A).$$

*Proof.* Associating to any  $t$ -torsion space  $(M, \varphi)$  the hermitian space  $(M, \partial_M \circ \varphi)$  preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic  $t$ -torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W: W_{\text{tors}}(A[t]) \rightarrow W(A).$$

To find a preimage  $(M, \varphi)$  of a space  $(M, \alpha)$  over  $A$  consider  $M$  as an  $A[t]$ -module annihilated by  $t$  and replace  $\alpha: M \rightarrow M^*$  by  $\varphi = \partial_M^{-1} \circ \alpha$ .  $\square$

## 5. THE WITT GROUP OF EXTENDED SPACES

Let  $W'(A[t, t^{-1}])$  be the group defined in the introduction.

THEOREM 5.1. *Let  $A$  be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi: W(A) \oplus W(A) \rightarrow W'(A[t, t^{-1}])$$

*mapping  $(\xi, \eta)$  to  $\xi + t\eta$  is an isomorphism.*

*Proof.* The injectivity of  $\psi$  is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$\operatorname{Res}: W'(A[t, t^{-1}]) \rightarrow W(A)$$

*with the following properties:*

$R_1$ : *For any constant space  $\xi \in W(A) \subset W'(A[t, t^{-1}])$ ,  $\operatorname{Res}(\xi) = 0$ .*

$R_2$ : *For any constant space  $\xi \in W(A) \subset W'(A[t, t^{-1}])$ ,  $\operatorname{Res}(t \cdot \xi) = \xi$ .*

*Proof.* See Theorem 6.7.  $\square$

Assuming this proposition, suppose that for two elements  $\xi, \eta \in W(A)$  we have  $\xi + t \cdot \eta = 0$ . Then  $0 = \text{Res}(\xi + t \cdot \eta) = \eta$  and hence  $\xi = 0$ .

We now turn to the surjectivity of  $\psi$ . We have to show that every hermitian space  $(P, \alpha)$  over  $A[t, t^{-1}]$  with  $P = P_0[t, t^{-1}]$  is Witt equivalent to a space of the form  $(Q_0[t, t^{-1}], \alpha_0) \perp (Q_1[t, t^{-1}], t\alpha_1)$ . Let  $P_1$  be a projective  $A$ -module such that  $P_0 \oplus P_1 = A^n$  for some  $n$ . Replacing  $(P, \alpha)$  by

$$(P_0[t, t^{-1}], \alpha) \perp (P_0[t, t^{-1}], -\alpha(1)) \perp H(P_1[t, t^{-1}]),$$

we may assume that  $P_0$  is free. Replacing  $\alpha$  by  $t^{2N}\alpha$  with a suitable  $N$ , we may also assume that  $\alpha$  maps  $P_0[t]$  into  $P_0^*[t]$ . By Lemma 3.2 we are reduced to the case where  $\alpha = \alpha_0 + t\alpha_1$  for some  $\epsilon$ -hermitian maps  $\alpha_0, \alpha_1: P_0 \rightarrow P_0^*$ .

LEMMA 5.3. *If, for a constant matrix  $\beta$ ,*

$$\alpha = 1 + (t - 1)\beta \in \text{GL}_n(A[t, t^{-1}]) \cap \text{M}_n(A[t]),$$

*then there exists an  $N$  such that  $(1 - \beta)^N \beta^N = 0$ .*

*Proof.* This is Corollary 2.4 of [2]. For the convenience of the reader we reprove it here.

Writing the inverse of  $\alpha$  as a Laurent polynomial and equating coefficients in the identity

$$1 = \alpha\alpha^{-1} = (1 - \beta + t\beta)(\gamma_{-q}t^{-q} + \cdots + \gamma_{-1}t^{-1} + \gamma_0 + \gamma_1t + \cdots + \gamma_pt^p)$$

we get

$$\begin{aligned} (1 - \beta)\gamma_{-q} &= 0, \quad (1 - \beta)\gamma_{-q+1} + \beta\gamma_{-q} = 0, \quad \dots, \\ (1 - \beta)\gamma_{-1} + \beta\gamma_{-2} &= 0, \quad (1 - \beta)\gamma_0 + \beta\gamma_{-1} = 1 \end{aligned}$$

and

$$(1 - \beta)\gamma_1 + \beta\gamma_0 = 0, \quad \dots, \quad (1 - \beta)\gamma_p + \beta\gamma_{p-1} = 0, \quad \beta\gamma_p = 0.$$

From the first line we get  $(1 - \beta)^q\gamma_{-1} = 0$ , from the third  $\beta^{p+1}\gamma_0 = 0$  and then from the middle one  $\beta^{p+1}(1 - \beta)^q = 0$ .  $\square$

We put  $\beta = \alpha(1)^{-1}\alpha_1: P_0 \rightarrow P_0$ , so that

$$\alpha(1)^{-1}\alpha = 1 + (t - 1)\beta.$$

We will repeatedly use the fact that  $\beta$  is adjoint with respect to  $\alpha, \alpha(1), \alpha_0, \alpha_1$ , by which we mean that  $\alpha\beta = \beta^*\alpha$ , and so on. The same clearly holds for any polynomial in  $\beta$  with integral coefficients.

By Lemma 5.3 we can find an integer  $N$  such that  $\beta^N(1 - \beta)^N = 0$ . Denoting by  $\mathbf{Z}[\beta]$  the subring of  $\text{End}_A(P_0)$  generated by  $\beta$  we can write  $\mathbf{Z}[\beta] = \mathbf{Z}[\beta]e \times \mathbf{Z}[\beta](1 - e)$ , where  $e$  is an idempotent of the form  $\beta + \nu$  and  $\nu$  is a nilpotent matrix. Note that  $e$  and  $\nu$  are polynomials in  $\beta$  and therefore they commute with  $\beta$  and with each other. If we decompose  $P_0$  as  $eP_0 + (1 - e)P_0$  and represent  $A$ -linear endomorphisms of  $P_0$  as  $2 \times 2$  block matrices, we have

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 + \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}$$

and

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \epsilon\alpha_{12}^* & \alpha_{22} \end{pmatrix} (1 + (t - 1)\beta).$$

Computing the product we see that the condition  $\alpha^* = \epsilon\alpha$  implies that

$$\alpha_{12}(1 - \nu_2) = -\nu_1^* \alpha_{12}, \quad \alpha_{11}^* = \epsilon\alpha_{11} \quad \text{and} \quad \alpha_{22}^* = \epsilon\alpha_{22}.$$

From this we immediately deduce

$$\alpha_{12}(1 - \nu_2)^k = (-\nu_1^*)^k \alpha_{12}$$

for any natural integer  $k$ . Since  $\nu_1$  and  $\nu_2$  are nilpotent, this implies that  $\alpha_{12} = 0$ . Thus  $\alpha$  is of the form

$$\begin{pmatrix} \alpha_{11}t(1 + \nu_1) - \alpha_{11}\nu_1 & 0 \\ 0 & \alpha_{22}(1 + (t - 1)\nu_2) \end{pmatrix}$$

and  $(P_0[t, t^{-1}], \alpha)$  splits as a hermitian space.

Since  $\alpha$ ,  $\alpha_{11}$  and  $\alpha_{22}$  are symmetric, evaluating the above matrix at  $t = 1$  we see that

$$\alpha_{11}\nu_1 = \nu_1^* \alpha_{11} \quad \text{and} \quad \alpha_{22}\nu_1 = \nu_2^* \alpha_{22}.$$

The first block can be written as

$$\sigma_1 = \alpha_{11}t(1 + \nu_1 - t^{-1}\nu_1) = \alpha_{11}t(1 + (1 - t^{-1})\nu_1).$$

Since  $(1 - t^{-1})\nu_1$  is nilpotent, the formal power series

$$\tau_1 = (1 + (1 - t^{-1})\nu_1)^{-1/2} = \sum \binom{-1/2}{k} ((1 - t^{-1})\nu_1)^k$$

is a Laurent polynomial and we can replace the first block by  $\tau_1^* \sigma_1 \tau_1 = \alpha_{11}t$ . Similarly, the power series

$$\tau_2 = (1 + (t - 1)\nu_2)^{-1/2} = \sum \binom{-1/2}{k} ((t - 1)\nu_2)^k$$

is a Laurent polynomial and we can replace the second block by  $\tau_2^* \sigma_2 \tau_2 = \alpha_{22}$ .

This shows that

$$(P_0[t, t^{-1}], \alpha) \simeq (P_0e[t, t^{-1}], t\alpha_{11}) \perp (P_0(1 - e)[t, t^{-1}], \alpha_{22}),$$

thus proving the surjectivity of  $\psi$ .  $\square$