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PROPOSITION 4.7. *The isomorphisms*

$$\partial_M\colon \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \to \mathrm{Hom}_A(M,A)$$

*induce a surjective homomorphism*

$$\partial^W\colon W_{tors}(A[t]) \to W(A)\,.$$

*Proof.* Associating to any $t$-torsion space $(M,\varphi)$ the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic $t$-torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W\colon W_{tors}(A[t]) \to W(A)\,.$$

To find a preimage $(M,\varphi)$ of a space $(M,\alpha)$ over $A$ consider $M$ as an $A[t]$-module annihilated by $t$ and replace $\alpha\colon M \to M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. □

## 5. THE WITT GROUP OF EXTENDED SPACES

Let $W'(A[t,t^{-1}])$ be the group defined in the introduction.

THEOREM 5.1. *Let $A$ be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi\colon W(A) \oplus W(A) \to W'(A[t,t^{-1}])$$

*mapping $(\xi,\eta)$ to $\xi + t\eta$ is an isomorphism.*

*Proof.* The injectivity of $\psi$ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$Res\colon W'(A[t,t^{-1}]) \to W(A)$$

*with the following properties:*

$R_1$ : *For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(\xi) = 0$.*

$R_2$ : *For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(t \cdot \xi) = \xi$.*

*Proof.* See Theorem 6.7. □

Assuming this proposition, suppose that for two elements $\xi, \eta \in W(A)$ we have $\xi + t\cdot\eta = 0$. Then $0 = Res(\xi + t\cdot\eta) = \eta$ and hence $\xi = 0$.

We now turn to the surjectivity of $\psi$. We have to show that every hermitian space $(P, \alpha)$ over $A[t, t^{-1}]$ with $P = P_0[t, t^{-1}]$ is Witt equivalent to a space of the form $(Q_0[t, t^{-1}], \alpha_0) \perp (Q_1[t, t^{-1}], t\alpha_1)$. Let $P_1$ be a projective $A$-module such that $P_0 \oplus P_1 = A^n$ for some $n$. Replacing $(P, \alpha)$ by

$$(P_0[t, t^{-1}], \alpha) \perp (P_0[t, t^{-1}], -\alpha(1)) \perp H(P_1[t, t^{-1}]),$$

we may assume that $P_0$ is free. Replacing $\alpha$ by $t^{2N}\alpha$ with a suitable $N$, we may also assume that $\alpha$ maps $P_0[t]$ into $P_0^*[t]$. By Lemma 3.2 we are reduced to the case where $\alpha = \alpha_0 + t\alpha_1$ for some $\epsilon$-hermitian maps $\alpha_0, \alpha_1 \colon P_0 \to P_0^*$.

LEMMA 5.3. *If, for a constant matrix $\beta$,*

$$\alpha = 1 + (t-1)\beta \in \mathrm{GL}_n(A[t, t^{-1}]) \cap \mathrm{M}_n(A[t]),$$

*then there exists an $N$ such that $(1-\beta)^N\beta^N = 0$.*

*Proof.* This is Corollary 2.4 of [2]. For the convenience of the reader we reprove it here.

Writing the inverse of $\alpha$ as a Laurent polynomial and equating coefficients in the identity

$$1 = \alpha\alpha^{-1} = (1 - \beta + t\beta)(\gamma_{-q}t^{-q} + \cdots + \gamma_{-1}t^{-1} + \gamma_0 + \gamma_1 t + \cdots + \gamma_p t^p)$$

we get

$$(1-\beta)\gamma_{-q} = 0,\ (1-\beta)\gamma_{-q+1} + \beta\gamma_{-q} = 0,\ \ldots,$$
$$(1-\beta)\gamma_{-1} + \beta\gamma_{-2} = 0,\ (1-\beta)\gamma_0 + \beta\gamma_{-1} = 1$$

and

$$(1-\beta)\gamma_1 + \beta\gamma_0 = 0,\ \ldots,\ (1-\beta)\gamma_p + \beta\gamma_{p-1} = 0,\ \beta\gamma_p = 0.$$

From the first line we get $(1-\beta)^q\gamma_{-1} = 0$, from the third $\beta^{p+1}\gamma_0 = 0$ and then from the middle one $\beta^{p+1}(1-\beta)^q = 0$. □

We put $\beta = \alpha(1)^{-1}\alpha_1 \colon P_0 \to P_0$, so that

$$\alpha(1)^{-1}\alpha = 1 + (t-1)\beta.$$

We will repeatedly use the fact that $\beta$ is adjoint with respect to $\alpha, \alpha(1), \alpha_0, \alpha_1$, by which we mean that $\alpha\beta = \beta^*\alpha$, and so on. The same clearly holds for any polynomial in $\beta$ with integral coefficients.

By Lemma 5.3 we can find an integer $N$ such that $\beta^N(1-\beta)^N = 0$. Denoting by $\mathbf{Z}[\beta]$ the subring of $\mathrm{End}_A(P_0)$ generated by $\beta$ we can write $\mathbf{Z}[\beta] = \mathbf{Z}[\beta]e \times \mathbf{Z}[\beta](1-e)$, where $e$ is an idempotent of the form $\beta + \nu$ and $\nu$ is a nilpotent matrix. Note that $e$ and $\nu$ are polynomials in $\beta$ and therefore they commute with $\beta$ and with each other. If we decompose $P_0$ as $eP_0 + (1-e)P_0$ and represent $A$-linear endomorphisms of $P_0$ as $2\times 2$ block matrices, we have

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1+\nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}$$

and

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \epsilon\alpha_{12}^* & \alpha_{22} \end{pmatrix} (1 + (t-1)\beta).$$

Computing the product we see that the condition $\alpha^* = \epsilon\alpha$ implies that

$$\alpha_{12}(1-\nu_2) = -\nu_1^*\alpha_{12}, \qquad \alpha_{11}^* = \epsilon\alpha_{11} \quad \text{and} \quad \alpha_{22}^* = \epsilon\alpha_{22}.$$

From this we immediately deduce

$$\alpha_{12}(1-\nu_2)^k = (-\nu_1^*)^k\alpha_{12}$$

for any natural integer $k$. Since $\nu_1$ and $\nu_2$ are nilpotent, this implies that $\alpha_{12} = 0$. Thus $\alpha$ is of the form

$$\begin{pmatrix} \alpha_{11}t(1+\nu_1) - \alpha_{11}\nu_1 & 0 \\ 0 & \alpha_{22}(1+(t-1)\nu_2) \end{pmatrix}$$

and $(P_0[t,t^{-1}], \alpha)$ splits as a hermitian space.

Since $\alpha$, $\alpha_{11}$ and $\alpha_{22}$ are symmetric, evaluating the above matrix at $t=1$ we see that

$$\alpha_{11}\nu_1 = \nu_1^*\alpha_{11} \quad \text{and} \quad \alpha_{22}\nu_1 = \nu_2^*\alpha_{22}.$$

The first block can be written as

$$\sigma_1 = \alpha_{11}t(1+\nu_1 - t^{-1}\nu_1) = \alpha_{11}t(1+(1-t^{-1})\nu_1).$$

Since $(1-t^{-1})\nu_1$ is nilpotent, the formal power series

$$\tau_1 = (1+(1-t^{-1})\nu_1)^{-1/2} = \sum \binom{-1/2}{k}((1-t^{-1})\nu_1)^k$$

is a Laurent polynomial and we can replace the first block by $\tau_1^*\sigma_1\tau_1 = \alpha_{11}t$. Similarly, the power series

$$\tau_2 = (1+(t-1)\nu_2)^{-1/2} = \sum \binom{-1/2}{k}((t-1)\nu_2)^k$$

is a Laurent polynomial and we can replace the second block by $\tau_2^*\sigma_2\tau_2 = \alpha_{22}$.

This shows that

$$(P_0[t,t^{-1}], \alpha) \simeq (P_0e[t,t^{-1}], t\alpha_{11}) \perp (P_0(1-e)[t,t^{-1}], \alpha_{22}),$$

thus proving the surjectivity of $\psi$. □