Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 46 (2000)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE SATURATION CONJECTURE (AFTER A. KNUTSON AND T. TAO)

Autor: BUCH, Anders Skovsted / Fulton, William

Kapitel: 5. Remarks and questions

DOI: https://doi.org/10.5169/seals-64794

Fix a generic functional ω on $\mathbf{R}^{H-B}$ which maps a hive to a linear combination with positive coefficients of the labels at non-border hive vertices. For each $b \in \rho(C)$, let $\ell(b)$ be the unique hive in $\rho^{-1}(b) \cap C$ where ω is maximal. Then $\ell: \rho(C) \to C$ is a continuous piece-wise linear map [11, § 1]. Notice that since w has positive coefficients, $\ell(b)$ has no increasable subsets.

We want to prove that the labels of $\ell(b)$ are $\mathbf{Z}$-linear combinations of the labels of b. In particular $\ell(b)$ is an integral hive if b is integral. For a regular border $b \in \rho(C)$, Proposition 1 implies that the flatspaces of $\ell(b)$ consist of small triangles and rhombi; by Proposition 2 this implies that all labels of $\ell(b)$ are $\mathbf{Z}$-linear combinations of the labels of b. Finally, since the regular borders are dense in each maximal subcone of $\rho(C)$ where ℓ is linear, ℓ must be integrally defined everywhere. □

5. Remarks and questions

Knutson and Tao's proof of the saturation conjecture implies that Klyachko's inequalities for T_n can be produced by a simple recursive algorithm, which uses the inequalities for T_k, $1 \leq k \leq n-1$ ([9], [10], [12], [6]). A triple of partitions (λ, μ, ν) with $|\nu| = |\lambda| + |\mu|$ is in T_n if and only if

$$\sum_{i=1}^{k} \nu_{\gamma_i+k+1-i} \leq \sum_{i=1}^{k} \lambda_{\alpha_i+k+1-i} + \sum_{i=1}^{k} \mu_{\beta_i+k+1-i}$$

for all triples $(\alpha, \beta, \gamma) \in T_k$ with $\gamma_1 \leq n-k$. Another important consequence is Horn's conjecture, which says that the same inequalities describe which sets of eigenvalues can arise from two Hermitian matrices and their sum [8].

P. Belkale has shown that the inequality produced by a triple (α, β, γ) with Littlewood-Richardson coefficient $c^{\gamma}_{\alpha\beta} \geq 2$ follows from the other inequalities. Knutson, Tao, and Woodward have announced a proof that the remaining inequalities are independent, i.e. they describe the facets of the cone $\rho(C)$. Their proof uses an interesting operation of overlaying two hives, which is defined in terms of Knutson and Tao's honeycomb model [10].

These results have made it very interesting to determine which triples (λ, μ, ν) have coefficient $c^{\nu}_{\lambda\mu}$ equal to one. Fulton has conjectured that this is equivalent to $c^{N\nu}_{N\lambda,N\mu}$ being one for any $N \in \mathbf{N}$. This has been verified in all cases with $N|\nu| \leq 68$. (Recently Knutson and Tao have reported that they can prove this as well.)

For $n = 3$ it is easy to show that a triple of partitions has Littlewood-Richardson coefficient one if and only if it corresponds to a point on the boundary of the cone $\rho(C)$. In general, Fulton's conjecture implies that the triples with coefficient one are exactly those corresponding to points in a collection of faces of $\rho(C)$. For $n \geq 3$ this means that all triples corresponding to interior points in $\rho(C)$ have coefficient at least two.

One approach for proving Fulton's conjecture is to show that if $b \in \rho(C) \cap \mathbf{Z}^B$, then any generic positive functional ω on $\mathbf{R}^{H-B}$ must be minimized (as well as maximized) at an integral hive in $\rho^{-1}(b) \cap C$. In fact, by Proposition 2 it is enough to prove:

If $b \in \rho(C)$ is a generic border and if a generic positive functional ω is minimized at $h \in \rho^{-1}(b) \cap C$, then the flatspaces of h consist of small triangles and rhombi.

Part of proving this is to specify when a border b is generic. We believe the statement is true if b avoids finitely many hyperplanes in $\mathbf{R}^B$.

The Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ have the following natural generalization. Given decreasing sequences of integers ν, and $\lambda(1), \dots, \lambda(r)$, let $c^{\nu}_{\lambda(1),\dots,\lambda(r)}$ denote the multiplicity of V_ν in the holomorphic representation $V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(r)}$. When $\nu = (0, \dots, 0)$, this specializes to the symmetric Littlewood-Richardson coefficient $c_{\lambda(1),\dots,\lambda(r)}$ which is the dimension of the $\mathrm{GL}_n(\mathbf{C})$-invariant subspace of $V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(r)}$. Postnikov and Zelevinsky have pointed out that the saturation conjecture as stated in the introduction implies a similar result for these generalized coefficients, i.e.

$$(5.1) \qquad c^{\nu}_{\lambda(1),\dots,\lambda(r)} \neq 0 \iff c^{N\nu}_{N\lambda(1),\dots,N\lambda(r)} \neq 0\,.$$

Knutson has shown us that, by combining several hive triangles, one obtains a polytope whose integral points count these more general coefficients. This gives rise to another proof of (5.1).

In [3] other generalized Littlewood-Richardson coefficients related to quiver varieties are described. A different generalization related to Hecke algebras is defined in [7], and quantum Littlewood-Richardson coefficients are studied in [2]. It would be very interesting if these coefficients can be realized as the number of integral points in some polytopes.