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4. A LIE ALGEBRA REPRESENTATION

Let M be a projective R -module of rank two. Let $G = \text{Aut}_R(M)$ and let $\mathfrak{g} = \text{End}_R(M)$ viewed as a Lie algebra over R .

The group G acts on the right on $\text{Sym}_R(M^*)$ by algebra automorphisms via

$$(F\sigma)(\mathbf{x}) = F(\sigma\mathbf{x})$$

for $F \in \text{Sym}_R(M^*)$ and $\sigma \in G$. Taking the formal derivative at the origin of the associated map

$$G \rightarrow \text{Aut}_{R\text{-alg}}(\text{Sym}_R(M^*))$$

we get a representation of Lie algebras

$$(22) \quad \rho: \mathfrak{g} \longrightarrow \text{Der}_R(\text{Sym}_R(M^*)).$$

The action of G preserves the homogeneous components $\text{Sym}_R^n(M^*)$ and also the submodule $S^n(M^*)$ of Gaussian forms. The same is true for the Lie algebra action of \mathfrak{g} .

We shall compute the action of \mathfrak{g} on $S^n(M^*)$ explicitly:

LEMMA 4.1. *Let $F \in S^n(M^*)$ and let T be the associated n -linear form. Then*

$$\rho(g)(F)(\mathbf{x}) = nT(g\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$$

for all $g \in \mathfrak{g}$.

Proof. To compute the derivative of $G \rightarrow \text{Aut}_R(S^n(M^*))$, we extend the scalars to the “dual numbers” $R[\epsilon]/(\epsilon^2)$. Using the symmetry of T we have

$$F((1 + g\epsilon)\mathbf{x}) = F(\mathbf{x}) + nT(g\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})\epsilon,$$

which proves our assertion. \square

Let C/R be a quadratic algebra in the sense of Section 2 and let M be an invertible C -module. Then we have a natural map $C \rightarrow \text{End}_R(M)$ and we can restrict the representation ρ to C . Note that when R is a field and C is an étale quadratic algebra then the image of C is a Cartan subalgebra \mathfrak{h}_C of \mathfrak{g} .

Comparing (22) with equation (21), we see that the C -module structure on $S_C^3(M^*)$ is related to the Lie algebra action by

$$(23) \quad cF = \frac{1}{3}\rho(c)F.$$

We will make this explicit in a special case that we need:

LEMMA 4.2. Let $F \in S^3(M^*)$ be a binary cubic form over a field K of characteristic not 2 or 3. Let q_F be its determining form, and $C = C^+(q_F)$ its even Clifford algebra. Let x_1, x_2 be coordinates on the vector space M with respect to a basis $\mathbf{m}_1, \mathbf{m}_2$. Let

$$\tau = \mathbf{m}_1 \mathbf{m}_2 - \mathbf{m}_2 \mathbf{m}_1 \in C = C^+(q_F).$$

Note that $\tau^2 = D$ is the discriminant of q_F . Then

$$\rho(\tau) = \frac{\partial q_F}{\partial x_2} \frac{\partial}{\partial x_1} - \frac{\partial q_F}{\partial x_1} \frac{\partial}{\partial x_2},$$

acting on forms of any degree.

Proof. As we have seen,

$$q_F(x_1 \mathbf{m}_1 + x_2 \mathbf{m}_2) = Px_1^2 + Qx_1x_2 + Rx_2^2,$$

where $P = a_1^2 - a_0a_2$, $Q = a_1a_2 - a_0a_3$, and $R = a_2^2 - a_1a_3$. By direct computation in the Clifford algebra C , we see that

$$\tau \mathbf{m}_1 = Q\mathbf{m}_1 - 2P\mathbf{m}_2$$

$$\tau \mathbf{m}_2 = 2R\mathbf{m}_1 - Q\mathbf{m}_2.$$

Since $\rho(c)$ is a derivation of $\text{Sym}_R(M^*)$, we have

$$\rho(c) = \rho(c)(x_1) \frac{\partial}{\partial x_1} + \rho(c)(x_2) \frac{\partial}{\partial x_2}.$$

Thus $\tau(x_1 \mathbf{m}_1 + x_2 \mathbf{m}_2) = (Qx_1 + 2Rx_2)\mathbf{m}_1 - (2Px_1 + Qx_2)\mathbf{m}_2$, which gives $\rho(\tau)(x_1) = \partial q_F / \partial x_2$ and $\rho(\tau)(x_2) = -\partial q_F / \partial x_1$. \square

COROLLARY 4.3. $\rho(\tau)q_F = 0$ and

$$(24) \quad \begin{aligned} \rho(\tau)F &= \begin{vmatrix} \partial F / \partial x_1 & \partial F / \partial x_2 \\ \partial q_F / \partial x_1 & \partial q_F / \partial x_2 \end{vmatrix} \\ &= 3G_F, \end{aligned}$$

where G_F is as in (5).

REMARK 4.4. If we further assume that C is an étale algebra, then as we have remarked, ρ maps C onto a Cartan subalgebra of $\text{End}_K(M) \sim \mathfrak{gl}(2, K)$. This algebra decomposes as

$$\mathfrak{h}_C = \mathfrak{z} \oplus \mathfrak{h}'_C$$

where the first factor is the center, consisting of scalar matrices, and the second factor is the intersection $\mathfrak{h}_C \cap \mathfrak{sl}(2, K)$, consisting of matrices of trace 0. As the formulas in the proof of the preceding lemma show that τ acts on M with trace 0, we see that $\mathfrak{h}'_C = K\tau$.

THEOREM 4.5. *Let C/R be a quadratic algebra such that $C \otimes K$ is étale over K . Let M be a projective rank-one C -module and let $F \in S^3(M^*)$ be such that the determining mapping q_F is not 0. Then the following conditions are equivalent:*

- (a) F is a C -form
- (b) $(M, q_F, \mathcal{D}(M))$ is of type C
- (c) $\rho(c)\rho(\bar{c})F = 9n(c)F$ for all $c \in C$.

Proof. (a) \Rightarrow (b). If T is the trilinear form attached to F , then, using the symmetry of $T(c\mathbf{x}, \mathbf{y}, \mathbf{z})$, we have

$$\begin{aligned} q_F(c\mathbf{x}) &= \wedge^2 T(c\mathbf{x}, -, -) \\ &= \wedge^2 (T(\mathbf{x}, c-, -)) \\ &= n(c) \wedge^2 (T(\mathbf{x}, -, -)) \\ &= n(c)q_F(\mathbf{x}), \end{aligned}$$

which proves the claim. In fact, this implication does not depend on $C \otimes K$ being étale.

It is enough to prove the theorem for the case where $R = K$ is a separably closed field. We can assume in this case $C = K[\sigma]$ with σ satisfying $\sigma^2 = 1$. We will make these assumptions for the rest of the proof.

(b) \Rightarrow (c). Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis of M over K with $\sigma\mathbf{m}_1 = \mathbf{m}_1$ and $\sigma\mathbf{m}_2 = -\mathbf{m}_2$. With respect to this basis, the form q_F , being of type C , must have the shape

$$q_F(\mathbf{x}) = \alpha x_1 x_2,$$

where $\alpha \neq 0$. To see that this is so, note that because q_F is of type C , we have $q_F(\sigma\mathbf{m}_1) = n(\sigma)q_F(\mathbf{m}_1) = -q_F(\mathbf{m}_1)$, which shows that $q_F(\mathbf{m}_1) = 0$. One sees similarly that $q_F(\mathbf{m}_2) = 0$. Then the coefficients of $F(\mathbf{x}) = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3$ satisfy the relations: $a_1^2 - a_0 a_2 = 0$, $a_1 a_2 - a_0 a_3 = \alpha$ and $a_2^2 - a_1 a_3 = 0$. Since $\alpha \neq 0$, it follows at once that $a_1 = a_2 = 0$, so F is of the form $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$. Since $q_F \neq 0$ (in fact nondegenerate under the étaleness hypothesis), the algebra C can be identified with the even Clifford algebra $C^+(M, q_F, \mathcal{D}(M))$ by Proposition 2.8. Under that identification we have $\tau = \alpha\sigma$, where τ is defined as in Lemma 4.2. From that lemma we get $\rho(\sigma) = x_1 \partial / \partial x_1 - x_2 \partial / \partial x_2$, which can be seen directly, since both sides agree on x_1, x_2 . Hence $\rho(\sigma)(x_1^{3-i} x_2^i) = (3 - 2i)x_1^{3-i} x_2^i$. In particular, for $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$ we have

$$\rho(\sigma)\rho(\bar{\sigma})F = -\rho(\sigma)^2 F = -9F = 9n(\sigma)F.$$

The more general identity $\rho(c)\rho(\bar{c})F = 9n(c)F$ for $c \in C$ follows from this particular case by noting that, from Lemma 4.1, $\rho(1)F = 3F$.

(c) \Rightarrow (a). Suppose that $\rho(\sigma)^2F = 9F$. Then F must have the form $F = \lambda x_1^3 + \mu x_2^3$. This is because, as we saw in the discussion above, the monomials $x_1^{3-i}x_2^i$ are eigenvectors for the operator $\rho(\sigma)^2$ with eigenvalue $(3-2i)^2$, which equals 9 only for $i = 0$ and $i = 3$. Hence the associated trilinear form is $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 + \mu x_2 y_2 z_2$. Thus $T(\sigma \mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 - \mu x_2 y_2 z_2$, which is visibly symmetric in $\mathbf{x}, \mathbf{y}, \mathbf{z}$. \square

REMARK 4.6. It is interesting to notice that the syzygy (6) can be recovered from Part (c) of Theorem 4.5. Assume for simplicity that $R = K$ is a field and C is an étale K -algebra. Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis of M . Let $\tau = \mathbf{m}_1 \mathbf{m}_2 - \mathbf{m}_2 \mathbf{m}_1 \in C = C^+(q_F)$ as in Lemma 4.2. As we noted in Remark 4.4, τ generates the trace 0 part of the Cartan subalgebra defined by C . Using the derivation property and Corollary 4.3, we see $\rho(\tau)(G_F^2 - DF^2) = (2/3)(\rho(\tau)^2F - 9DF)G_F$. From the above theorem, $\rho(\tau)^2F = 9DF$, so this is 0. On the other hand, $\rho(\tau)q_F = 0$, also by Corollary 4.3, which implies that $\rho(\tau)q_F^3 = 0$. Hence both q_F^3 and $G_F^2 - DF^2$ lie in the subspace on weight 0 (for the action of the Cartan subalgebra $\mathfrak{h}'_C \subset \mathfrak{sl}(2, K)$) of $S^6(M^*)$. As $S^6(M^*)$ is an irreducible representation of $\mathfrak{sl}(2, K)$, this is one-dimensional. Hence q_F^3 and $G_F^2 - DF^2$ differ by a constant multiple. A priori, this constant could depend on F (e.g., D). That this is not so can be seen by noting that both sides are of the same degree in the coefficients of F .

COROLLARY 4.7. *Let M be a projective R -module of rank 2, and let $F \in S^3(M^*)$.*

- (i) *Let $C = C^+(M, q_F, \mathcal{D}(M))$ and suppose that $C \otimes K$ is étale, and that q_F is primitive. Then F is a C -form.*
- (ii) *If F is a C -form for a quadratic R -algebra C and $(M, q_F, \mathcal{D}(M))$ is primitive, then $C \cong C^+(M, q_F, \mathcal{D}(M))$.*

Proof. (i) By Proposition 2.8, $(M, q_F, \mathcal{D}(M))$ is of type C . We conclude by Theorem 4.5.

(ii) If F is a C -form, then by Theorem 4.5, the quadratic mapping $(M, q_F, \mathcal{D}(M))$ is type C . But assuming furthermore that $(M, q_F, \mathcal{D}(M))$ is primitive, we see that C is isomorphic with $C^+(M, q_F, \mathcal{D}(M))$ by Proposition 2.8. \square

LEMMA 4.8. *Suppose that $C \otimes K$ is étale over K and let (M, F) and (M', F') be cubic C -forms. Assume that the determining mappings $q_F, q_{F'}$ are nonzero. Then every R -linear isomorphism $f: (M, F) \rightarrow (M', F')$ is either C -linear or C -sesquilinear.*

Proof. The map f will induce an isomorphism of determining quadratic mappings of type C . We conclude by Proposition 2.3. \square

5. STRUCTURE OF THE CUBIC C -FORMS

We shall describe below the C -module structure of $S_C^3(M^*)$ and the corresponding C -isomorphism classes.

THEOREM 5.1. *Let M be a rank-one projective C -module. For each $\phi \in \text{Hom}_C(M_C^{\otimes 3}, C^*)$ we define a cubic form by $F_\phi(\mathbf{x}) = \phi(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})(1)$. Then*

- (i) *The correspondence $\phi \mapsto F_\phi$ is an isomorphism of C -modules $\text{Hom}_C(M_C^{\otimes 3}, C^*) \rightarrow S_C^3(M^*)$.*
- (ii) *The determining mapping q_{F_ϕ} is primitive if and only if ϕ is an isomorphism.*
- (iii) *Two cubic C -forms F and F_1 on M are equivalent over C if and only if there exists $c \in C^\times$ such that $F_1 = c^3 F$.*

Proof. (i) This is a restatement of Proposition 3.7. The map $\phi \mapsto F_\phi$ is a C -isomorphism by definition of the structure of C -module on $S_C^3(M^*)$ in Section 3.

(ii) It is enough to prove our assertion locally, so we assume that M is free over C . Write $M = C\mathbf{m}$ for some $\mathbf{m} \in M$. Let $\lambda = \phi(\mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m})$. Then we have $\phi(x\mathbf{m} \otimes y\mathbf{m} \otimes z\mathbf{m}) = \lambda(xyz)$. Let $\beta(y\mathbf{m}, z\mathbf{m}) = \lambda(yz)$ and observe that λ is a basis of C^* over C if and only if the symmetric bilinear form β is unimodular. We have

$$\begin{aligned} q_{F_\phi}(x\mathbf{m}) &= n(x)q_{F_\phi}(\mathbf{m}) \\ &= n(x) \wedge^2 \beta. \end{aligned}$$

It follows from this equality that q_{F_ϕ} is primitive if and only if β is unimodular, that is, if and only if ϕ is an isomorphism.