1. Introduction

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1. Introduction

The Radon transform on a manifold X associates to a function u on this manifold its integrals $Ru(y)$ over a given family Y of submanifolds y (equipped with suitable measures). One of the main problems of integral geometry is to recover u from Ru by means of an explicit inversion formula. The dual Radon transform R^* then enters the picture in a natural way: it maps functions on Y into functions on X, by integrating (with respect to a suitable measure) over all submanifolds $y \in Y$ which contain a given point $x \in X$.

Here we assume that a Lie group G acts transitively on both X and Y, so that they are homogeneous spaces $X = G/K$, $Y = G/H$ where K, H are Lie subgroups of G; besides, K will be compact throughout the paper. Our main examples for X will be Riemannian symmetric spaces of the noncompact type, often assumed to have rank one (hyperbolic spaces). For Y they will be a family of totally geodesic submanifolds of X, or the family of horocycles.

We first look for a left inverse of R of the following form

$$(*) \qquad u(x) = DR^*Ru(x)\,,$$

where D is some operator acting on functions on X. In all known examples D is an integro-differential operator, sometimes even differential. The purpose of the present paper is to emphasize three simple ideas leading to such results (or related to them), sometimes hidden under long calculations dealing with some specific example. As a benefit we can unify several proofs from the literature, and obtain some generalizations.

a. The first idea stems from Proposition 3 (Section 3.1): R^*R is always a *convolution operator* on X, by a K-invariant measure S. Besides, S can easily be written down explicitly on rank one examples (Propositions 4 and 5). The problem is thus to find a convolution inverse D to S. We study it in Section 4 for noncompact isotropic spaces (i.e. all Euclidean or hyperbolic spaces), looking for D as a polynomial of the Laplace-Beltrami operator of X with given fundamental solution S. This can be done for the Radon transform on even-dimensional totally geodesic submanifolds (with an additional assumption, see Theorem 8), or on horocycles of odd-dimensional hyperbolic spaces (Theorem 9).

Another natural approach is to seek the convolution inverse D by means of K-invariant harmonic analysis on X. We discuss this in Section 5 for the totally geodesic Radon transform on hyperbolic spaces. Unfortunately it seems difficult to find D explicitly by this method, except under the asssumptions of Theorem 8 (proved by simpler tools) or for the case of $X = H^n(\mathbf{R})$ (already solved by Berenstein and Tarabusi [1]).

b. The second idea goes back to Johann Radon himself, and will be developed here in full generality. If we replace R^* by the *shifted dual transform* R_t^*, obtained by integrating over all submanifolds y at distance t (in some sense) from a point x, we can prove new inversion formulas for R. More precisely for $X = G/K$, $Y = G/H$ we consider (Section 6.1)

$$Ru(gH) = \int_H u(ghK)\, dh\,, \quad R_t^* v(gK) = \int_K v(gktH)\, dk\,,$$

where g, t are elements of G, u is a function on X and v on Y. Of course $R_t^* = R^*$ when t is the identity. It is then quite elementary to observe (Section 6.2) that an inversion formula of R at the origin x_o for K-invariant u, say $u(x_o) = \langle T_{(y)}, Ru(y)\rangle$, implies the following new result

$$(**) \qquad u(x) = \langle T_{(t)}, R_t^* Ru(x)\rangle\,,$$

for arbitrary u and x. The notation $T_{(t)}$ means that the operator T now acts on the shift variable t, instead of x as in $(*)$. Applying this method to the horocycle transform on Riemannian symmetric spaces of the noncompact type, we obtain a new proof of Helgason's inversion formulas (Theorem 13 and Corollary 20). In Theorem 14 the same method is applied to the totally geodesic transform, thus extending to all classical hyperbolic spaces known results for the real ones.

c. It is now an intriguing question to compare the results $(*)$ and $(**)$ of methods a and b. For the 2-dimensional totally geodesic tranform on $X = H^3(\mathbf{R})$, Helgason ([10]) obtained a curious "*amusing formula*" by equating the right-hand sides of $(*)$ and $(**)$. In Sections 6.4 and 6.5 we give direct proofs of such formulas, for the Laplace operator first (Proposition 16), then for general invariant differential operators (Theorem 17).

The content of these results is easily understood on the example of the Radon transform on all hyperplanes of $X = \mathbf{R}^{2k+1}$ (see Section 6.4 for more details). Here the inversion formulas $(*)$, resp. $(**)$, are

$$Cu(x) = L_x^k R^* Ru(x)\ , \text{ resp. } Cu(x) = \partial_t^{2k} R_t^* Ru(x)\big|_{t=0}\,,$$

where C is a constant factor and L is the Euclidean Laplacian. Passing from one to the other is thus an immediate consequence of the *wave equation*

$$L_x R_t^* v(x) = \partial_t^2 R_t^* v(x)$$

for $v = Ru$ and all x and t. In Proposition 16 and Theorem 17 we construct solutions of some generalized wave equations, some of them only valid when t is the identity (but this suffices for our purpose). Such results may have independent interest, providing explicit solutions of certain multitemporal wave equations by means of shifted dual Radon transforms, which appear as integrals of elementary "plane" waves (Proposition 19, for horocycles).

One last remark: explicit inversion formulas for the totally geodesic Radon transform seem rather difficult to obtain, and most of those in the literature are only given for spaces of constant curvature. We obtain here some results for $X = H^n(\mathbf{F})$, with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, provided that the tangent spaces to the geodesic submanifolds under consideration are $\mathbf{F}$-vector spaces (Section 4.3 **c**, Theorem 14, Proposition 16). This seems to be the simplest case after $\mathbf{R}^n$ and $H^n(\mathbf{R})$.

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NOTATIONS

a. GENERAL NOTATIONS. As usual $\mathbf{R}, \mathbf{C}, \mathbf{H}$ respectively denote the fields of real numbers, complex numbers and quaternions. When considering vector spaces on $\mathbf{H}$, the scalars will act on the right.

If X is a (real C^∞) manifold, $C(X)$ is the space of complex-valued continuous functions on X, $C_c(X)$ the subspace of compactly supported functions and $\mathcal{D}(X)$ the subspace of compactly supported C^∞ functions; $\mathcal{D}'(X)$ is the space of distributions and $\mathcal{E}'(X)$ the subspace of compactly supported distributions.

If T is an operator (e.g. differential) on a space of functions on X, a notation like $T_{(x)} f(x,y)$ means that T acts on the variable x, not y.

If G is a (real) Lie group, let e, $\mathfrak{g}$, $\exp$, Ad, ad respectively denote its origin, Lie algebra, exponential mapping, adjoint representations of G and $\mathfrak{g}$.

When G acts on X, we shall write $g \cdot x$, or sometimes $\tau(g)x$ or even $\tau_X(g)x$, for the point obtained when $g \in G$ acts on the point $x \in X$. In particular, for $V \in \mathfrak{g}$, it is convenient to write $g \cdot V$ for $\mathrm{Ad}(g)V$. In this context, $\mathbf{D}(X)$ is the algebra of linear differential operators on X which commute to the action of G, and $\mathbf{D}(G)$ refers to the special case when G acts onto itself by left translations.

If X is a Riemannian manifold, $\Sigma(x, r)$ will denote the sphere with center $x \in X$ and radius $r \geq 0$. Also $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in the Euclidean space $\mathbf{R}^n$.

b. RIEMANNIAN HOMOGENEOUS SPACES. Let G be a Lie group, K a *compact* subgroup and $\mathfrak{g}$, $\mathfrak{k}$ their Lie algebras. The homogeneous manifold $X = G/K$ can be provided with a G-invariant Riemannian structure. Indeed a scalar product can be taken on $\mathfrak{g}$, invariant under the compact group $\mathrm{Ad}\,_G(K)$; then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, is a K-invariant (i.e. stable under $\mathrm{Ad}\,_G(K)$) vector subspace which can be identified with the tangent space to X at the origin $x_o = K$. Carrying by the action of G the K-invariant scalar product on $\mathfrak{p}$ we thus obtain a Riemannian structure on X, and elements of G are isometries.

We shall also consider $Y = G/H$, where H is another Lie subgroup of G.

c. RIEMANNIAN SYMMETRIC SPACES (see [8], Chap. IV or [15], Chap. XI for their basic properties). A special case of the previous one, they are the homogeneous spaces $X = G/K$, where G is a connected Lie group provided with an involutive automorphism θ and K is a compact subgroup which lies between the group of all fixed points of θ in G and its identity component. The differential of θ at e induces a Lie algebra automorphism of $\mathfrak{g}$ and the eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (same notations as before).

The exponential mapping of the symmetric space is $\mathrm{Exp} : \mathfrak{p} \to X$, related to $\exp : \mathfrak{g} \to G$ by $\mathrm{Exp}\, V = (\exp V)K$ for $V \in \mathfrak{p}$. The curve $\mathrm{Exp}\,\mathbf{R}V$ is the geodesic of X which is tangent to the vector V at the origin $x_o = K$.

d. RIEMANNIAN SYMMETRIC SPACES OF THE NONCOMPACT TYPE. Assuming further that G is a connected non compact real semisimple Lie group with finite center and K a maximal compact subgroup, we obtain the subclass of Riemannian symmetric spaces of the noncompact type, particularly interesting because of their rich (and well-known) structure arising from the theory of root systems. The map $\mathrm{Exp} : \mathfrak{p} \to X$ is then a global diffeomorphism onto.

We briefly recall some classical semisimple notations, as used for instance in Helgason's books. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Related to the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$ are the eigenspaces $\mathfrak{g}_\alpha$, the *Iwasawa decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the Lie algebra and $G = KAN$ for the group (unique decomposition of each element of G into a product of factors in the respective subgroups); the subgroups A, resp. N, of G are abelian, resp. nilpotent. The half sum of positive roots (counted with multiplicities) is a linear form ρ on $\mathfrak{a}$; we write $a^\rho = e^{\rho(\log a)}$ for $a \in A$. Let M, resp. M', denote the centralizer, resp. normalizer, of A in K. Then $W = M'/M$ is a finite group called the Weyl group.

Let y_o denote the orbit $N \cdot x_o \subset X$. The *horocycles* of X are the submanifolds $g \cdot y_o$, for $g \in G$. Since $g \cdot y_o = y_o$ (globally) if and only if $g \in MN$, the space of all horocycles is $Y = G/MN$.

e. ISOTROPIC RIEMANNIAN SYMMETRIC SPACES. A Riemannian manifold X is called *isotropic* if, for every $x \in X$ and every pair of unit tangent vectors V, W to X at x, there exists an isometry of X leaving x fixed and mapping V to W. The connected isotropic Riemannian manifolds are the Euclidean spaces $\mathbf{R}^n$, the *hyperbolic spaces* i.e. the Riemannian symmetric spaces of the noncompact type and of rank one ($\dim \mathfrak{a} = 1$), and their compact analogues, spheres and projective spaces. The compact spaces will not be considered in this paper, so that most of our examples will be taken from the list

$$\mathbf{R}^n,\ H^n(\mathbf{R}),\ H^n(\mathbf{C}),\ H^n(\mathbf{H}),\ H^{16}(\mathbf{O}).$$

Among them we shall often restrict ourselves to the *classical hyperbolic spaces* $H^n(\mathbf{F})$, with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$.

2. GEOMETRIC SETTING

2.1 DOUBLE FIBRATIONS OF HOMOGENEOUS SPACES

The general group-theoretic setting for Radon transforms, introduced by Helgason in the sixties, is motivated by the well-known example of points and hyperplanes in the Euclidean space $\mathbf{R}^n$. The set of points and the set of hyperplanes are both homogeneous spaces of the isometry group of $\mathbf{R}^n$, and it turns out that the fundamental "incidence" relation (a point x belongs to a hyperplane y), as well as the defining integral of the Radon transform, have simple expressions in terms of Lie groups and invariant measures. This observation suggests considering the following general situation.