2. Geometric setting

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We briefly recall some classical semisimple notations, as used for instance in Helgason's books. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Related to the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$ are the eigenspaces $\mathfrak{g}_\alpha$, the *Iwasawa decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the Lie algebra and $G = KAN$ for the group (unique decomposition of each element of G into a product of factors in the respective subgroups); the subgroups A, resp. N, of G are abelian, resp. nilpotent. The half sum of positive roots (counted with multiplicities) is a linear form ρ on $\mathfrak{a}$; we write $a^\rho = e^{\rho(\log a)}$ for $a \in A$. Let M, resp. M', denote the centralizer, resp. normalizer, of A in K. Then $W = M'/M$ is a finite group called the Weyl group.

Let y_o denote the orbit $N \cdot x_o \subset X$. The *horocycles* of X are the submanifolds $g \cdot y_o$, for $g \in G$. Since $g \cdot y_o = y_o$ (globally) if and only if $g \in MN$, the space of all horocycles is $Y = G/MN$.

e. Isotropic Riemannian symmetric spaces. A Riemannian manifold X is called *isotropic* if, for every $x \in X$ and every pair of unit tangent vectors V, W to X at x, there exists an isometry of X leaving x fixed and mapping V to W. The connected isotropic Riemannian manifolds are the Euclidean spaces $\mathbf{R}^n$, the *hyperbolic spaces* i.e. the Riemannian symmetric spaces of the noncompact type and of rank one ($\dim \mathfrak{a} = 1$), and their compact analogues, spheres and projective spaces. The compact spaces will not be considered in this paper, so that most of our examples will be taken from the list

$$\mathbf{R}^n,\ H^n(\mathbf{R}),\ H^n(\mathbf{C}),\ H^n(\mathbf{H}),\ H^{16}(\mathbf{O}).$$

Among them we shall often restrict ourselves to the *classical hyperbolic spaces* $H^n(\mathbf{F})$, with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$.

2. Geometric setting

2.1 Double fibrations of homogeneous spaces

The general group-theoretic setting for Radon transforms, introduced by Helgason in the sixties, is motivated by the well-known example of points and hyperplanes in the Euclidean space $\mathbf{R}^n$. The set of points and the set of hyperplanes are both homogeneous spaces of the isometry group of $\mathbf{R}^n$, and it turns out that the fundamental "incidence" relation (a point x belongs to a hyperplane y), as well as the defining integral of the Radon transform, have simple expressions in terms of Lie groups and invariant measures. This observation suggests considering the following general situation.

Let X and Y be two manifolds, *with given origins* $x_o \in X$ and $y_o \in Y$, and assume a real Lie group G *acts transitively on both manifolds* X and Y. Two elements $x \in X$ and $y \in Y$ are said to be *incident* if there exists some $g \in G$ such that $x = g \cdot x_o$ and $y = g \cdot y_o$. Roughly speaking, if we think of g as a motion, this means that x and y have the same relative position as the origins x_o and y_o.

A more convenient formulation is obtained in terms of the isotropy subgroups K, resp. H, of x_o, resp. y_o, in G. They are closed Lie subgroups of G, and the manifolds X, Y can be identified with the homogeneous spaces of left cosets G/K, G/H respectively; in particular we may write $x_o = K$, $y_o = H$, $g \cdot x_o = gK$, etc. The points $x = g'K \in X$ and $y = g''H \in Y$ are then incident if and only if there exists $g \in G$ such that $g'K = g \cdot x_o = gK$ and $g''H = g \cdot y_o = gH$, in other words if *the left cosets* $g'K$ *and* $g''H$, *as subsets of* G, *are not disjoint* (they meet at g).

Given $y = g''H$, we see that x is incident to y if and only if $x = g''hK$ for some $h \in H$. Given $x = g'K$, the point y is incident to x if and only if $y = g'kH$ for some $k \in K$.

In the above example X, resp. Y, is the set of points, resp. hyperplanes, of $\mathbf{R}^n$ and G is the group of all isometries. But hyperplanes can also be viewed as subsets of $X = \mathbf{R}^n$, and the incidence relation boils down to the familiar "the point x belongs to the hyperplane y" if and only if the chosen origin x_o belongs to the chosen origin y_o. Lemma 1 below extends this fact to Riemannian manifolds. More general incidence relations can be considered, however, and will be helpful in Section 6.

Clearly, the group G acts transitively on the subset Z of $X \times Y$ consisting of all incident couples $(x, y) = (g \cdot x_o, g \cdot y_o)$, with $K \cap H$ as the isotropy subgroup of the origin $(x_o, y_o) \in Z$. Thus $Z = G/(K \cap H)$ can be endowed with a structure of manifold, and the present setting can be summarized by the following *double fibration of homogeneous spaces*

$$\begin{array}{ccc} Z = G/(K \cap H) \subset X \times Y & & \\ \downarrow \qquad\qquad \searrow & & \\ X = G/K \qquad Y = G/H, & & \end{array}$$

where the arrows denote the natural projections.

Radon transforms can be studied with more general double fibrations of manifolds X, Y, Z (without groups), as introduced by Gel′fand et al. [4]. We refer to Guillemin and Sternberg ([6], p. 340, 370) for their basic properties; this theory has been developed in several papers by Boman, Quinto, and others.

2.2 Group-theoretic Radon transforms

Let G be a real Lie group and K a (closed) Lie sugbroup, equipped with left-invariant Haar measures dg, dk respectively. If the homogeneous space G/K admits a G-invariant measure $d(gK)$, the measures can then be normalized so that

$$\int_G f(g)\,dg = \int_{G/K} d(gK) \int_K f(gk)\,dk\,,$$

for any $f \in C_c(G)$. This applies in particular if K is compact (on invariant measures, see [9], Chap. I, § 1).

Throughout the paper G will be a Lie group, K a *compact* subgroup, and H a (closed) Lie subgroup of G. The Haar measure dk of K will be normalized by $\int_K dk = 1$.

Let u be a (complex-valued) function on $X = G/K$. Its *Radon transform* is the function Ru on $Y = G/H$ defined by

$$Ru(gH) = \int_H u(ghK)\,dh\,,$$

for $g \in G$, whenever this makes sense (e.g. if $u \in C_c(X)$). The left invariance of dh implies that the integral only depends on the left coset gH of g. Given $y = gH$ in $Y = G/H$, the value $Ru(y)$ is an integral of u over all x incident to y. A more precise statement can be given in the following important example.

Example. Let X be a connected Riemannian manifold, G a transitive Lie group of *isometries* of X and K the isotropy subgroup of some origin $x_o \in X$; then K is compact ([8], p. 204) and $X = G/K$. Let y_o be a given *closed submanifold of* X, *containing* x_o, and let Y be the set of all submanifolds $y = g \cdot y_o$ of X, with $g \in G$.

The set H of all $h \in G$ such that $h \cdot y_o = y_o$ (i.e. the submanifold y_o is globally invariant under h) is a closed Lie subgroup of G. Indeed if $h_n \in H$ converges to h in G, for any $x \in y_o$ the point $\lim h_n \cdot x = h \cdot x$ belongs to y_o; similarly $h^{-1} \cdot x \in y_o$, so that $h \cdot y_o = y_o$. Thus $Y = G/H$ can be endowed with a structure of manifold and we obtain a double fibration of homogeneous spaces.

The following lemma allows one to compute the Radon transform without knowing H explicitly.

LEMMA 1. *Keeping the notation of this example, assume furthermore that* $y_o = G' \cdot x_o$ *is a closed orbit of the origin* $x_o = K$ *under some Lie subgroup* G' *of* G.

Then $G' \subset H \subset G'K$ *and* $y_o = H \cdot x_o$. *The incidence relation between* $X = G/K$ *and* $Y = G/H$ *is simply* $x \in y$. *Besides, the left-invariant Haar measures* dh, dg' *of the groups* H, G' *can be normalized so that*

$$\begin{aligned} Ru(y) &= \int_H u(gh \cdot x_o)\, dh = \int_{G'} u(gg' \cdot x_o)\, dg' \\ &= \int_y u(x)\, dm_y(x), \end{aligned}$$

where dm_y *is the Riemannian measure induced by* X *on its submanifold* $y = g \cdot y_o$.

REMARK. The subgroup H can of course be strictly bigger than G'. This occurs for instance if y_o is a line in $X = \mathbf{R}^n$ and G' is the group of translations along this line, or a horocycle in a Riemannian symmetric space X of the noncompact type (for which $G' = N$ and $H = MN = NM$ in the usual semisimple notations).

Proof of Lemma 1. If $y_o = G' \cdot x_o$, then H obviously contains G' and it follows that

$$y_o = G' \cdot x_o \subset H \cdot x_o \subset y_o ,$$

whence $H \cdot x_o = G' \cdot x_o$ and $H \subset G'K$.

A point $x \in X$ is incident to $y = g \cdot y_o \in Y$ if and only if there exists $h \in H$ such that $x = gh \cdot x_o$, i.e. $x \in gH \cdot x_o = g \cdot y_o = y$.

An isometry g transforms the Riemannian measure of y_o into the Riemannian measure of $y = g \cdot y_o$, and it suffices to prove the integral formula for $g = e$. Now $y_o = H \cdot x_o$ can be identified to the homogeneous space $H/(H \cap K)$, and dm_{y_o} (which is invariant under all isometries of y_o) to an H-invariant measure on this space. The Haar measure dh can therefore be normalized so that the corresponding measure on $H/(H \cap K)$ satisfies

$$\begin{aligned} \int_{y_o} u(x)\, dm_{y_o}(x) &= \int_{H/(H\cap K)} u(h \cdot x_o)\, d(h(H \cap K)) \\ &= \int_H u(h \cdot x_o)\, dh = Ru(y_o). \end{aligned}$$

The proof is similar for $\int_{G'}$, whence the lemma. $\square$

Going back to general double fibrations, the *Radon dual transform* of a (continuous, say) function v on $Y = G/H$ is the function on $X = G/K$ defined by

$$R^*v(gK) = \int_K v(gkH)\, dk\,,$$

for $g \in G$, an integral of v over all y incident to $x = gK$. The word "dual" is of course motivated by the classical projective duality between points and hyperplanes in the basic example, but it also stems from the following proposition.

PROPOSITION 2. *Let $X = G/K$ with K compact, and assume that $Y = G/H$ has a G-invariant measure. Let $u \in C_c(X)$, $v \in C(Y)$. Then $Ru \in C_c(Y)$, $R^*v \in C(X)$ and*

$$\int_X u(x)\, R^*v(x)\, dx = \int_Y Ru(y)\, v(y)\, dy = \int_Z u(x)\, v(y)\, dz\,,$$

where dx, dy, dz are the respective G-invariant measures on X, Y and $Z = G/(K \cap H)$.

In the latter integral $u(x)v(y)$ is considered as a function of $z = (x,y)$ on Z (Section 2.1). We omit the proof, a classical exercise on invariant integrals (cf. [9], p. 144 and [11], p. 41); all groups are assumed unimodular there, but the proof only uses the invariant measures on the homogeneous spaces, thus extends to the present situation.

Proposition 2 allows a natural extension of the transforms R and R^* to distributions. Given $u \in \mathcal{E}'(X)$, the distribution $Ru \in \mathcal{E}'(Y)$ is defined by

$$\langle Ru, v\rangle = \langle u, R^*v\rangle\,,$$

for all test functions $v \in C^\infty(Y)$. Similarly, given $v \in \mathcal{D}'(Y)$, the distribution $R^*v \in \mathcal{D}'(X)$ is defined by

$$\langle R^*v, u\rangle = \langle v, Ru\rangle\,,$$

for all $u \in \mathcal{D}(X)$. Again we refer to Helgason ([11], p. 42) for details, based on the compactness of K. These definitions do extend the Radon integrals for functions, as Proposition 2 shows, when identifying a function u with the distribution $u(x)\, dx$, and similarly for v.