2.2 Group-theoretic Radon transforms

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Let G be a real Lie group and K a (closed) Lie sugbroup, equipped with left-invariant Haar measures dg, dk respectively. If the homogeneous space G/K admits a G-invariant measure $d(gK)$, the measures can then be normalized so that

$$\int_G f(g)\,dg = \int_{G/K} d(gK) \int_K f(gk)\,dk\,,$$

for any $f \in C_c(G)$. This applies in particular if K is compact (on invariant measures, see [9], Chap. I, § 1).

Throughout the paper G will be a Lie group, K a *compact* subgroup, and H a (closed) Lie subgroup of G. The Haar measure dk of K will be normalized by $\int_K dk = 1$.

Let u be a (complex-valued) function on $X = G/K$. Its *Radon transform* is the function Ru on $Y = G/H$ defined by

$$Ru(gH) = \int_H u(ghK)\,dh\,,$$

for $g \in G$, whenever this makes sense (e.g. if $u \in C_c(X)$). The left invariance of dh implies that the integral only depends on the left coset gH of g. Given $y = gH$ in $Y = G/H$, the value $Ru(y)$ is an integral of u over all x incident to y. A more precise statement can be given in the following important example.

Example. Let X be a connected Riemannian manifold, G a transitive Lie group of *isometries* of X and K the isotropy subgroup of some origin $x_o \in X$; then K is compact ([8], p. 204) and $X = G/K$. Let y_o be a given *closed submanifold of* X, *containing* x_o, and let Y be the set of all submanifolds $y = g \cdot y_o$ of X, with $g \in G$.

The set H of all $h \in G$ such that $h \cdot y_o = y_o$ (i.e. the submanifold y_o is globally invariant under h) is a closed Lie subgroup of G. Indeed if $h_n \in H$ converges to h in G, for any $x \in y_o$ the point $\lim h_n \cdot x = h \cdot x$ belongs to y_o; similarly $h^{-1} \cdot x \in y_o$, so that $h \cdot y_o = y_o$. Thus $Y = G/H$ can be endowed with a structure of manifold and we obtain a double fibration of homogeneous spaces.

The following lemma allows one to compute the Radon transform without knowing H explicitly.

LEMMA 1. *Keeping the notation of this example, assume furthermore that* $y_o = G' \cdot x_o$ *is a closed orbit of the origin* $x_o = K$ *under some Lie subgroup* G' *of* G.

Then $G' \subset H \subset G'K$ *and* $y_o = H \cdot x_o$. *The incidence relation between* $X = G/K$ *and* $Y = G/H$ *is simply* $x \in y$. *Besides, the left-invariant Haar measures* dh, dg' *of the groups* H, G' *can be normalized so that*

$$\begin{aligned} Ru(y) &= \int_H u(gh \cdot x_o)\, dh = \int_{G'} u(gg' \cdot x_o)\, dg' \\ &= \int_y u(x)\, dm_y(x)\,, \end{aligned}$$

where dm_y *is the Riemannian measure induced by* X *on its submanifold* $y = g \cdot y_o$.

REMARK. The subgroup H can of course be strictly bigger than G'. This occurs for instance if y_o is a line in $X = \mathbf{R}^n$ and G' is the group of translations along this line, or a horocycle in a Riemannian symmetric space X of the noncompact type (for which $G' = N$ and $H = MN = NM$ in the usual semisimple notations).

Proof of Lemma 1. If $y_o = G' \cdot x_o$, then H obviously contains G' and it follows that

$$y_o = G' \cdot x_o \subset H \cdot x_o \subset y_o\,,$$

whence $H \cdot x_o = G' \cdot x_o$ and $H \subset G'K$.

A point $x \in X$ is incident to $y = g \cdot y_o \in Y$ if and only if there exists $h \in H$ such that $x = gh \cdot x_o$, i.e. $x \in gH \cdot x_o = g \cdot y_o = y$.

An isometry g transforms the Riemannian measure of y_o into the Riemannian measure of $y = g \cdot y_o$, and it suffices to prove the integral formula for $g = e$. Now $y_o = H \cdot x_o$ can be identified to the homogeneous space $H/(H \cap K)$, and dm_{y_o} (which is invariant under all isometries of y_o) to an H-invariant measure on this space. The Haar measure dh can therefore be normalized so that the corresponding measure on $H/(H \cap K)$ satisfies

$$\begin{aligned} \int_{y_o} u(x)\, dm_{y_o}(x) &= \int_{H/(H\cap K)} u(h \cdot x_o)\, d(h(H \cap K)) \\ &= \int_H u(h \cdot x_o)\, dh = Ru(y_o)\,. \end{aligned}$$

The proof is similar for $\int_{G'}$, whence the lemma. $\square$

Going back to general double fibrations, the *Radon dual transform* of a (continuous, say) function v on $Y = G/H$ is the function on $X = G/K$ defined by

$$R^*v(gK) = \int_K v(gkH)\, dk\,,$$

for $g \in G$, an integral of v over all y incident to $x = gK$. The word "dual" is of course motivated by the classical projective duality between points and hyperplanes in the basic example, but it also stems from the following proposition.

PROPOSITION 2. *Let* $X = G/K$ *with* K *compact, and assume that* $Y = G/H$ *has a* G*-invariant measure. Let* $u \in C_c(X)$, $v \in C(Y)$. *Then* $Ru \in C_c(Y)$, $R^*v \in C(X)$ *and*

$$\int_X u(x)\, R^*v(x)\, dx = \int_Y Ru(y)\, v(y)\, dy = \int_Z u(x)\, v(y)\, dz\,,$$

where dx, dy, dz *are the respective* G*-invariant measures on* X, Y *and* $Z = G/(K \cap H)$.

In the latter integral $u(x)v(y)$ is considered as a function of $z = (x, y)$ on Z (Section 2.1). We omit the proof, a classical exercise on invariant integrals (cf. [9], p. 144 and [11], p. 41); all groups are assumed unimodular there, but the proof only uses the invariant measures on the homogeneous spaces, thus extends to the present situation.

Proposition 2 allows a natural extension of the transforms R and R^* to distributions. Given $u \in \mathcal{E}'(X)$, the distribution $Ru \in \mathcal{E}'(Y)$ is defined by

$$\langle Ru, v\rangle = \langle u, R^*v\rangle\,,$$

for all test functions $v \in C^\infty(Y)$. Similarly, given $v \in \mathcal{D}'(Y)$, the distribution $R^*v \in \mathcal{D}'(X)$ is defined by

$$\langle R^*v, u\rangle = \langle v, Ru\rangle\,,$$

for all $u \in \mathcal{D}(X)$. Again we refer to Helgason ([11], p. 42) for details, based on the compactness of K. These definitions do extend the Radon integrals for functions, as Proposition 2 shows, when identifying a function u with the distribution $u(x)\, dx$, and similarly for v.