4.2 An inversion formula

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The compact cases (projective spaces) might be dealt with similarly. One should then normalize H by $\alpha(H) = i$ and replace ε by i. Integrals with respect to r should run from 0 to the diameter ℓ of X, i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

4.2 An inversion formula

The G-invariant differential operators on an isotropic space X are the polynomials of its Laplace-Beltrami operator L ([9], p. 288). In order to invert the d-geodesic Radon transform on X, Section 3.2 suggests looking for a polynomial P such that the above distribution S is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on X defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^a (\cosh \varepsilon r)^b = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^b ,$$

where a and b are real constants and r is the distance from the origin x_o; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n f_{n-1,q} , \quad S(r) = \left(\omega_d/\omega_n\right) f_{d-n,q'-q}$$

with q, q', n and d as defined above.

Proposition 6. *Assume $\varepsilon = 0$ (Euclidean case), or $\varepsilon = 1$ and $b = 0$, or else $\varepsilon = 1$ and $b = 1 - q$ (hyperbolic cases). Then, for any integer $k \geq 1$, the function $f_{2k-n,b}$ defines a K-invariant distribution $F_{2k-n,b}$ on X such that*

$$P_k(L) F_{2k-n,b} = \omega_n 2^{k-1}(k-1)!\,(2-n)(4-n)\cdots(2k-n)\,\delta ,$$

where δ is the Dirac distribution at the origin x_o and P_k is the polynomial

$$P_k(x) = \prod_{j=1}^{k} \left(x + \varepsilon^2(n-2j-b)(2j+b+q-1)\right) .$$

Remark. The case $b = 0$, $n = 2k+2$ was given by Schimming and Schlichtkrull [17], Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.

Proof. By [9], p. 313 the radial part of L is

$$\begin{aligned}\Delta &= \partial_r^2 + \frac{A'(r)}{A(r)}\partial_r = A(r)^{-1} \circ \partial_r \circ A(r) \circ \partial_r \\ &= \partial_r^2 + ((n-1)\varepsilon \coth \varepsilon r + q\varepsilon \tanh \varepsilon r)\,\partial_r \\ &= \partial_r^2 + (p\varepsilon \coth \varepsilon r + 2q\varepsilon \coth 2\varepsilon r)\,\partial_r\,.\end{aligned}$$

The proof of the proposition breaks up into a few easy facts. First we have, for any $a, b \in \mathbf{R}$, the following equality of functions of $r > 0$:

$$(6)\quad \big(\Delta - \varepsilon^2(a+b)(a+n+b+q-1)\big) f_{a,b} = a(a+n-2)f_{a-2,b} - \varepsilon^2 b(b+q-1)f_{a,b-2}\,,$$

which is immediate from $\Delta f = A^{-1}(Af')'$ and the identities

$$f'_{a,b} = af_{a-1,b+1} + \varepsilon^2 bf_{a+1,b-1}\,, \quad f_{a,b} = f_{a,b-2} + \varepsilon^2 f_{a+2,b-2}\,.$$

LEMMA 7. *For $a+n \geq 2$, $\varepsilon = 0$ or 1, the locally integrable function $f_{a,b}$ defines a K-invariant distribution $F_{a,b}$ on X such that*

$$\big(L - \varepsilon^2(a+b)(a+n+b+q-1)\big)\, F_{a,b} = \begin{cases} a(a+n-2)F_{a-2,b} - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n > 2 \\ \omega_n a\,\delta - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n = 2 \end{cases}$$

(equality of distributions on X).

EXAMPLE. Taking $b = 0$, resp. $b = 1-q$, the lemma provides the following fundamental solutions (which coincide for $q = 1$)

$$\big(L + \varepsilon^2(n-2)(q+1)\big)\left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{2-n} = \omega_n(2-n)\delta$$

$$\big(L + 2\varepsilon^2(n+q-3)\big)\left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{2-n} (\cosh \varepsilon r)^{1-q} = \omega_n(2-n)\delta\,.$$

In the flat case $\varepsilon = 0$ they both reduce to $Lr^{2-n} = \omega_n(2-n)\delta$, a classical result for $\mathbf{R}^n$.

Proof of Lemma 7. Due to the K-invariance of $f_{a,b}$ and L we need only consider K-invariant test functions $u \in \mathcal{D}(X)$. The integral

$$\int_X f_{a,b} \cdot u\,dx = \int_0^\infty f_{a,b}(r)u(r)A(r)\,dr = \omega_n \int_0^\infty f_{a+n-1,b+q}(r)u(r)\,dr\,,$$

absolutely convergent if $a+n>0$, defines a distribution $F_{a,b}$ on X. In view of the symmetry and K-invariance of the Laplace operator we have

$$\begin{aligned}\langle LF_{a,b},u\rangle &= \langle F_{a,b},Lu\rangle\\ &= \int_0^\infty f_{a,b}(r)\Delta u(r)A(r)\,dr = \int_0^\infty f_{a,b}(Au')'\,dr\\ &= \left(Af'_{a,b}u\right)(0) - \left(Af_{a,b}u'\right)(0) + \int_0^\infty (Af'_{a,b})'u\,dr\,.\end{aligned}$$

If $a+n>2$ the function $Af_{a,b}$ vanishes to order $a+n-1$ at the origin, and $Af'_{a,b}$ to order $a+n-2$. Since $u(r)$ is smooth (this notation stands for $u(\mathrm{Exp}\,rH)$ with $\|H\|=1$), it follows that

$$\langle LF_{a,b},u\rangle = \int_0^\infty \Delta f_{a,b}(r)u(r)A(r)\,dr\,,$$

whence the result by (6).

The case $a+n=2$ is similar, in view of $(Af'_{a,b})(0)=\omega_n a$. □

Proposition 6 now follows easily: letting

$$L_a = L - \varepsilon^2(a+b)(a+n+b+q-1)$$

we have, by Lemma 7,

$$L_aF_{a,b} = \begin{cases} a(a+n-2)\,F_{a-2,b} & \text{if } a+n>2\\ \omega_n a\,\delta & \text{if } a+n=2\end{cases}$$

whenever $\varepsilon^2 b(b+q-1)=0$. Since

$$P_k(L) = L_{2-n}L_{4-n}\cdots L_{2k-n}\,,$$

the proposition follows by induction on k. □

THEOREM 8. *The d-geodesic Radon transform on a n-dimensional non-compact Riemannian isotropic space X can be inverted by means of a polynomial of its Laplace-Beltrami operator L, under the following assumptions:*

(i) *d is even: $d=2k$ with $k\geq 1$;*

(ii) *$X=\mathbf{R}^n$, or $\dim\mathfrak{s}_{2\alpha}=\dim\mathfrak{p}_{2\alpha}$, or else $\dim\mathfrak{s}_{2\alpha}=1$.*

Then

$$Cu = P_k(L)R^*Ru\,,$$

for any $u\in\mathcal{D}(X)$, where P_k is the polynomial from Proposition 6 (with $\varepsilon=1$, $q=\dim\mathfrak{p}_{2\alpha}$ and $b+q=\dim\mathfrak{s}_{2\alpha}$ if X is hyperbolic, or $\varepsilon=0$ if $X=\mathbf{R}^n$) and

$$C = \omega_d(-1)^k 2^{k-1}(k-1)!(n-2)(n-4)\cdots(n-2k)\,.$$

Proof. By (5) one has $S = (\omega_d/\omega_n) f_{a,b}$, with $a = d - n$ and $b = \dim \mathfrak{s}_{2\alpha} - \dim \mathfrak{p}_{2\alpha} = q' - q$ (Section 4.1 **b**). The theorem follows from Proposition 6 and Section 3.2. $\square$

Theorem 8 encompasses Helgason's Theorems 4.5 and 4.17 in [9], Chapter I (with different normalizations from ours), as well as some generalizations (next section). See also Grinberg [5] for the case of projective spaces, where the polynomial P_k is related to representation theory.

4.3 Examples

According to assumption (ii), three types of totally geodesic Radon transforms can be inverted by Theorem 8. Putting aside the case of even-dimensional planes in the Euclidean space $X = \mathbf{R}^n$, we now describe some examples of the latter two.

The space $X = G/K$ is then one of the hyperbolic spaces, and the dual space Y consists of all geodesic submanifolds $g \cdot \operatorname{Exp} \mathfrak{s}$, $g \in G$, where $\mathfrak{s} \subset \mathfrak{p}$ is an even-dimensional Lie triple system. Let $\mathfrak{a} = \mathbf{R}H$ be any line in $\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ be the corresponding root space decomposition.

a. A simple example is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$, assuming $\mathfrak{p}_{2\alpha} \neq 0$. Classical bracket relations (e.g. [8], p. 335) imply that $\mathfrak{s}$ is a Lie triple system and, reading $\dim \mathfrak{p}_{2\alpha}$ from the classification of rank one spaces, $\dim \mathfrak{s}$ is 2, 4 or 8; here $\mathfrak{s}_\alpha = 0$ and $\mathfrak{s}_{2\alpha} = \mathfrak{p}_{2\alpha}$.

b. Another example is $\mathfrak{s} = \mathfrak{p}_\alpha$, assuming this space is even-dimensional. Bracket relations show $\mathfrak{s}$ is a Lie triple system. To obtain compatible root space decompositions of $\mathfrak{s}$ and $\mathfrak{p}$ we replace H by an $H' \in \mathfrak{s}$, whence the new root space decompositions with respect to $\mathfrak{a}' = \mathbf{R}H'$

$$\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{p}'_\alpha \oplus \mathfrak{p}'_{2\alpha}, \quad \mathfrak{s} = \mathfrak{a}' \oplus \mathfrak{s}'_\alpha \oplus \mathfrak{s}'_{2\alpha}.$$

It follows again from the classification that $\mathfrak{p}'_{2\alpha}$ and $\mathfrak{s}'_{2\alpha}$ have the same dimension in all cases, therefore coincide (Helgason [7], p. 171, or [9], p. 168). This example is motivated by the Radon transform on antipodal manifolds of compact symmetric spaces of rank one (loc. cit.).

c. Totally geodesic transform on classical hyperbolic spaces. Let $X = H^m(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1, m; \mathbf{F})$, $K = U(1; \mathbf{F}) \times U(m; \mathbf{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices