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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

Kapitel: 4.3 Examples

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Proof. By (5) one has $S = (\omega_d/\omega_n) f_{a,b}$, with $a = d - n$ and $b = \dim \mathfrak{s}_{2\alpha} - \dim \mathfrak{p}_{2\alpha} = q' - q$ (Section 4.1 **b**). The theorem follows from Proposition 6 and Section 3.2. □

Theorem 8 encompasses Helgason's Theorems 4.5 and 4.17 in [9], Chapter I (with different normalizations from ours), as well as some generalizations (next section). See also Grinberg [5] for the case of projective spaces, where the polynomial P_k is related to representation theory.

4.3 EXAMPLES

According to assumption (ii), three types of totally geodesic Radon transforms can be inverted by Theorem 8. Putting aside the case of even-dimensional planes in the Euclidean space $X = \mathbf{R}^n$, we now describe some examples of the latter two.

The space $X = G/K$ is then one of the hyperbolic spaces, and the dual space Y consists of all geodesic submanifolds $g \cdot \operatorname{Exp} \mathfrak{s}$, $g \in G$, where $\mathfrak{s} \subset \mathfrak{p}$ is an even-dimensional Lie triple system. Let $\mathfrak{a} = \mathbf{R}H$ be any line in $\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ be the corresponding root space decomposition.

a. A simple example is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$, assuming $\mathfrak{p}_{2\alpha} \neq 0$. Classical bracket relations (e.g. [8], p. 335) imply that $\mathfrak{s}$ is a Lie triple system and, reading $\dim \mathfrak{p}_{2\alpha}$ from the classification of rank one spaces, $\dim \mathfrak{s}$ is 2, 4 or 8; here $\mathfrak{s}_\alpha = 0$ and $\mathfrak{s}_{2\alpha} = \mathfrak{p}_{2\alpha}$.

b. Another example is $\mathfrak{s} = \mathfrak{p}_\alpha$, assuming this space is even-dimensional. Bracket relations show $\mathfrak{s}$ is a Lie triple system. To obtain compatible root space decompositions of $\mathfrak{s}$ and $\mathfrak{p}$ we replace H by an $H' \in \mathfrak{s}$, whence the new root space decompositions with respect to $\mathfrak{a}' = \mathbf{R}H'$

$$\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{p}'_\alpha \oplus \mathfrak{p}'_{2\alpha}, \quad \mathfrak{s} = \mathfrak{a}' \oplus \mathfrak{s}'_\alpha \oplus \mathfrak{s}'_{2\alpha}.$$

It follows again from the classification that $\mathfrak{p}'_{2\alpha}$ and $\mathfrak{s}'_{2\alpha}$ have the same dimension in all cases, therefore coincide (Helgason [7], p. 171, or [9], p. 168). This example is motivated by the Radon transform on antipodal manifolds of compact symmetric spaces of rank one (loc. cit.).

c. TOTALLY GEODESIC TRANSFORM ON CLASSICAL HYPERBOLIC SPACES. Let $X = H^m(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1, m; \mathbf{F})$, $K = U(1; \mathbf{F}) \times U(m; \mathbf{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices

$$V = \begin{pmatrix} 0 & \overline{V}_1 & \cdots & \overline{V}_m \\ V_1 & & & \\ \vdots & & (0) & \\ V_m & & & \end{pmatrix}, \quad V_i \in \mathbf{F},$$

can be identified with $\mathbf{F}^m$.

Let $\overline{V} \cdot W = \sum_{i=1}^m \overline{V}_i W_i$. For $U, V, W \in \mathfrak{p} = \mathbf{F}^m$, easy computations lead to

$$(7) \qquad [U,[V,W]] = U\left(\overline{V} \cdot W - \overline{W} \cdot V\right) - V(\overline{W} \cdot U) + W(\overline{V} \cdot U)$$

($\mathbf{F}^m$ being considered as a $\mathbf{F}$-vector space, with scalars acting on the right). It follows that *any* $\mathbf{F}$*-subspace* $\mathfrak{s}$ *of* $\mathfrak{p}$ *is a Lie triple system.* Similarly, the natural inclusions $\mathbf{R}^m \subset \mathbf{C}^m \subset \mathbf{H}^m$ show that any $\mathbf{R}$-subspace of $\mathfrak{p} \cap \mathbf{R}^m$, or any $\mathbf{C}$-subspace of $\mathfrak{p} \cap \mathbf{C}^m$, is a Lie triple system.

Let $H \neq 0$ be an element of $\mathfrak{p}$. The eigenspaces of $(\operatorname{ad} H)^2$ can be obtained from (7), whence the decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}, \quad \mathfrak{a} = \mathbf{R}H,$$
$$\mathfrak{p}_\alpha = \{V \in \mathfrak{p} \mid \overline{H} \cdot V = 0\}, \quad \mathfrak{p}_{2\alpha} = \{H\lambda \mid \lambda \in \mathbf{F},\ \lambda + \overline{\lambda} = 0\},$$

with respective eigenvalues 0, $\overline{H} \cdot H$ and $4\left(\overline{H} \cdot H\right)$. A similar decomposition holds for $\mathfrak{s}$, if H is chosen in $\mathfrak{s}$. The spaces $\mathfrak{a} \oplus \mathfrak{p}_{2\alpha} = H\mathbf{F}$ and $\mathfrak{p}_\alpha$ are $\mathbf{F}$-subspaces of $\mathfrak{p}$, therefore Lie triple systems (as mentioned in a and b above). More generally, Theorem 8 applies to the following four families of totally geodesic submanifolds $\operatorname{Exp} \mathfrak{s}$; all superscripts in the table are real dimensions, with k, l, m strictly positive integers.

X	$\dim \mathfrak{p}_\alpha$	$\dim \mathfrak{p}_{2\alpha}$	$\mathfrak{s}$	$\dim \mathfrak{s}_\alpha$	$\dim \mathfrak{s}_{2\alpha}$	$y_o = \operatorname{Exp} \mathfrak{s}$
$H^m(\mathbf{R})$	$m-1$	0	(1)	$2k-1$	0	$H^{2k}(\mathbf{R})$
$H^{2m}(\mathbf{C})$	$2m-2$	1	(2)	$2k-2$	1	$H^{2k}(\mathbf{C})$
$H^{4m}(\mathbf{H})$	$4m-4$	3	(3)	$2k-2$	1	$H^{2k}(\mathbf{C})$
$H^{4m}(\mathbf{H})$	$4m-4$	3	(4)	$4l-4$	3	$H^{4l}(\mathbf{H})$

Case (1): $\mathfrak{s}$ is any $\mathbf{R}$-subspace of $\mathfrak{p} = \mathbf{R}^m$, with $\dim_{\mathbf{R}} \mathfrak{s} = 2k \leq m$.
Case (2): $\mathfrak{s}$ is any $\mathbf{C}$-subspace of $\mathfrak{p} = \mathbf{C}^m$, with $\dim_{\mathbf{C}} \mathfrak{s} = k \leq m$.
Case (3): $\mathfrak{s}$ is any $\mathbf{C}$-subspace of $\mathbf{C}^m \subset \mathfrak{p} = \mathbf{H}^m$, with $\dim_{\mathbf{C}} \mathfrak{s} = k \leq m$.
Case (4): $\mathfrak{s}$ is any $\mathbf{H}$-subspace of $\mathfrak{p} = \mathbf{H}^m$, with $\dim_{\mathbf{H}} \mathfrak{s} = l \leq m$.

d. HOROCYCLE TRANSFORM ON REAL HYPERBOLIC SPACES. Proposition 6 also applies to this case, because of the similarity between the functions S obtained in Propositions 4 and 5.

Following the same steps as for geodesic submanifolds, one can find a polynomial of the Laplacian with fundamental solution S (case $q = 0$ in Proposition 5). Indeed $S(r)$ is now, up to a constant factor, $f_{-1,2-n}(r/2)$ in the notation of Section 4.2 with $\varepsilon = 1$. Let

$$\Delta_{p,q} = \partial_r^2 + (p \coth r + 2q \coth 2r)\,\partial_r$$

be the radial part of the Laplacian and $g(r) = f(r/2)$. Then

$$4\left(\Delta_{p,0}\, g\right)(r) = \left(\Delta_{0,p} f\right)\left(r/2\right)\ ;$$

note that the roles of p and q have been interchanged. The next theorem now follows from Propositions 5 and 6, with $n = 2k+1$, $\varepsilon = 1$ and $b = 1 - p = 2 - n$.

THEOREM 9 (Helgason). *The horocycle Radon transform on the odd-dimensional hyperbolic space $X = H^{2k+1}(\mathbf{R})$, $k \geq 1$, is inverted by*

$$Cu = Q_k(L) R^* R u\,,$$

where $u \in \mathcal{D}(X)$, L is the Laplace-Beltrami operator of X,

$$C = \left(-\frac{\pi}{2}\right)^k \frac{(2k-1)!}{(k-1)!}\,, \quad Q_k(x) = \prod_{j=1}^{k} (x + j(2k-j))\ .$$

In [11], p. 210, the normalization of the Riemannian metric on X differs from ours.

The result extends to the horocycle transform on a Riemannian symmetric space $X = G/K$ of the noncompact type, provided that the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below). The spaces $H^{2k+1}(\mathbf{R})$ in Theorem 9 are the rank one spaces among those.