

Zeitschrift: L'Enseignement Mathématique
Band: 47 (2001)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH
Kapitel: 6.5 MULTITEMPORAL WAVES
Autor: Rouvière, François
DOI: <https://doi.org/10.5169/seals-65436>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 07.10.2024

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

scalar product $(T, V) = \operatorname{Re} \sum \bar{T}_i V_i$ on \mathfrak{p} we have $(T, V\lambda) = (T\bar{\lambda}, V)$, $\lambda \in \mathbf{F}$, therefore \mathfrak{s}^\perp is a \mathbf{F} -subspace of \mathfrak{p} .

An element k of $K \cap H$ is characterized by $k \in K$ and $k \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$, i.e. $k \cdot \mathfrak{s} = \mathfrak{s}$ (adjoint action). Let n', d' be the respective dimensions of \mathfrak{p} and \mathfrak{s} as \mathbf{F} -vector spaces. Taking a \mathbf{F} -basis of \mathfrak{p} according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, it follows that

$$K = U(1; \mathbf{F}) \times U(n'; \mathbf{F}), \quad K \cap H = U(1; \mathbf{F}) \times U(d'; \mathbf{F}) \times U(n' - d'; \mathbf{F}).$$

But $U(n' - d'; \mathbf{F})$ acts transitively on the unit sphere of $\mathbf{F}^{n' - d'}$, which implies our claim.

If $T, T' \in \mathfrak{s}^\perp$ are two unit vectors, there exists $k_o \in K \cap H$ such that $k_o \cdot T = T'$. Thus

$$\begin{aligned} R_{\exp tT'}^* v(gK) &= \int_K v(gkk_o \exp(tT)k_o^{-1}H) dk \\ &= \int_K v(gk \exp(tT)H) dk = R_{\exp tT}^* v(gK). \end{aligned}$$

In particular $R_{\exp tT}^* v$ is an even function of t .

Going back to (23), we now take as (X_j) an orthonormal \mathbf{R} -basis of \mathfrak{p} according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. The $n - d$ basis vectors in \mathfrak{s}^\perp give the same contribution to the right hand side, whereas the d vectors in \mathfrak{s} generate one parameters subgroups of H and give no contribution; indeed $\exp tV \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$ for $V \in \mathfrak{s}$, since \mathfrak{s} is a Lie triple system by Section 4.3 c. This completes the proof. \square

6.5 MULTITEMPORAL WAVES

We shall now deal with general invariant differential operators. As before G is a Lie group, H a closed subgroup, K a compact subgroup, and $X = G/K$, $Y = G/H$. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ be the respective Lie algebras, and \mathfrak{t} a vector subspace of \mathfrak{g} such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let K_1, \dots, K_p be a basis of \mathfrak{k} , complemented by $H_1, \dots, H_q \in \mathfrak{h}$ so that the K_i 's and H_j 's are a basis of $\mathfrak{k} + \mathfrak{h}$, and let T_1, \dots, T_r be a basis of \mathfrak{t} . We shall use the same notations for the corresponding left-invariant vector fields on G , e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with $f \in C^\infty(G)$, $g \in G$, $s \in \mathbf{R}$. We denote by $\mathbf{D}(G)$ the algebra of all left invariant differential operators on G , by $\mathbf{D}(G)^K$ the subalgebra of right

K -invariant operators and by $\mathbf{D}(X)$ the algebra of G -invariant differential operators on X . For $s = (s_1, \dots, s_r) \in \mathbf{R}^r$, let

$$t(s) = \exp s_1 T_1 \cdots \exp s_r T_r.$$

We recall that, for $g, t \in G$,

$$R_t^* v(gK) = \int_K v(gktH) dk.$$

THEOREM 17. *Let G be a Lie group, H, K Lie subgroups, with K compact and $X = G/K$, $Y = G/H$.*

(i) *For any $P \in \mathbf{D}(X)$ there exists $Q(\partial)$, a constant coefficients differential operator on \mathbf{R}^r , with $\text{order}(Q) \leq \text{order}(P)$, such that for any $v \in C^\infty(Y)$, $x \in X$,*

$$(25) \quad PR^* v(x) = Q(\partial_s) R_{t(s)}^* v(x) \Big|_{s=0}.$$

(ii) *Assume furthermore that \mathfrak{t} is a Lie subalgebra of \mathfrak{g} with $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{h}$, and let T denote the connected Lie subgroup of G with Lie algebra \mathfrak{t} . Then for any $P \in \mathbf{D}(X)$ there exists a right-invariant differential operator Q on T , with $\text{order}(Q) \leq \text{order}(P)$, such that*

$$(26) \quad P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x)$$

for $v \in C^\infty(Y)$; here $P_{(x)}$ acts on the variable $x \in X$ and $Q_{(t)}$ acts on $t \in T$.

Thus $R_t^* v(x)$, as a function of $(x, t) \in X \times T$, solves the generalized “multitemporal” wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables $(x, s) \in X \times \mathbf{R}^r$ at the time $s = 0$.

Proof. In order to work on G rather than on its homogeneous spaces, we define $w(g) = v(gH)$ and, for $g, t \in G$,

$$(27) \quad F(g, t) = (R_t^* v)(gK) = \int_K w(gkt) dk,$$

so that $F(gk, k'th) = F(g, t)$ for any $k, k' \in K$, $h \in H$, and

$$F(g, e) = (R^* v)(gK) = \int_K w(gk) dk.$$

Let $P \in \mathbf{D}(X)$ be given. Since K is compact the coset space $X = G/K$ is reductive and there exists $D \in \mathbf{D}(G)^K$ such that ([9], p. 285)

$$(28) \quad (Pf)(gK) = D_{(g)}(f(gK))$$

for $f \in C^\infty(X)$, $g \in G$.

To transfer derivatives from g to t we observe that, by the invariance of D under left translation by gk and right translation by k ,

$$D_{(g)}w(gkt) = D_{(x)}w(gkxt)|_{x=e},$$

where g, x, t are variables in G . Integrating over K it follows that

$$(29) \quad D_{(g)}F(g, t) = D_{(x)}F(g, xt)|_{x=e},$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \dots K_p^{\beta_p} T_1^{\alpha_1} \dots T_r^{\alpha_r} H_1^{\gamma_1} \dots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of $\mathbf{D}(G)$. Setting apart the terms with $\beta = \gamma = 0$, we can thus write, for some $E_i, F_j \in \mathbf{D}(G)$ and some constant coefficients a_α ,

$$(30) \quad D = D' + \sum_{i=1}^p K_i E_i + \sum_{j=1}^q F_j H_j, \quad D' = \sum_{\alpha} a_{\alpha_1 \dots \alpha_r} T_1^{\alpha_1} \dots T_r^{\alpha_r}.$$

If we replace $D_{(x)}$ by (30) in (29), the second term $(K_i E_i)_{(x)} F(g, xt)|_{x=e}$ vanishes because $K_i \in \mathfrak{k}$ and $F(g, kxt) = F(g, t)$. In the third term the left invariant vector field $H_j \in \mathfrak{h}$ acts by

$$(H_j)_{(x)} F(g, xt) = \partial_s F(g, x \exp(sH_j)t)|_{s=0},$$

and this vanishes too whenever t normalizes H , because $F(g, xth) = F(g, xt)$.

Since $t = e$ in case (i), or $t \in T$ with $Ht = tH$ in case (ii), we finally obtain for both cases (in multi-index notation)

$$(31) \quad \begin{aligned} D_{(g)}F(g, t) &= D'_{(x)}F(g, xt)|_{x=e} \\ &= \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} F(g, (\exp s_1 T_1 \dots \exp s_r T_r)t)|_{s=0} \\ &= \left(\sum_{\alpha} a_{\alpha} \partial_s^{\alpha} \right) F(g, t(s)t)|_{s=0}. \end{aligned}$$

Let the operator Q be defined by

$$Qf(t) = \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} f(t(s)t)|_{s=0},$$

a right invariant differential operator on the group T in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii). \square