3.3 BARNER'S THEOREM FOR POLYGONS

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Let H' be a hyperplane close to H and transverse to S_d; assume, further, that H' contains no vertices. It is enough to show that H' cannot intersect S_d in more than $d-1$ points. On the one hand, H' cannot intersect all the edges of S_d. Or else, H' would separate all pairs of consecutive vertices, and this would contradict the choice of W. On the other hand, if the number of intersections of H' and S_d were greater than $d-1$, it would be equal to $d+1$. Indeed, for topological reasons, the parity of this intersection number is that of $d-1$. We obtain a contradiction, which proves the claim.

Finally, by Lemma 3.3, the intersection multiplicity of H with S_d is not less than $d-1$. □

A curious property of a simplex is that each of its d-tuples of vertices is a flattening.

LEMMA 3.10. *The simplex S_d has $d+1$ flattenings.*

Proof. The determinant (3.1) involves all $d+1$ vectors $\widetilde{V}_1, \dots, \widetilde{V}_{d+1}$. If d is odd then, according to (3.3), $\widetilde{V}_{d+2} = \widetilde{V}_1$, and we are reduced to the fact that a cyclic permutation of vectors changes the sign of the determinant. On the other hand, if d is even then $\widetilde{V}_{d+2} = -\widetilde{V}_1$, which also leads to a change of sign in (3.1). □

3.3 BARNER'S THEOREM FOR POLYGONS

Now we formulate the result which serves as the main technical tool in the proof of Theorems 2.2, 2.6 and 2.10. Recall that we consider generic polygons in $\mathbf{RP}^d$ with at least $d+1$ vertices.

THEOREM 3.11. *A strictly convex polygon in $\mathbf{RP}^d$ has at least $d+1$ flattenings.*

Proof. Induction on the number n of vertices.

Induction starts with $n = d+1$. Up to projective transformations, the unique strictly convex $(d+1)$-gon is the simplex S_d. Indeed, every generic $(d+1)$-tuple of points in $\mathbf{RP}^d$ can be taken into any other one by a projective transformation. Therefore, all generic broken lines with d edges are projectively equivalent. It remains for us to connect the last point with the first one, and there are exactly two ways of doing this. One yields a contractible polygon, and the other a non-contractible one. One of these polygons is S_d, while the other one cannot be strictly convex, since its intersection number with a hyperplane does not have the same parity as $d-1$. The base for induction is then provided by Lemma 3.10.

Let P be a strictly convex $(n+1)$-gon with vertices $V_1, \dots, V_{n+1}$. Delete V_{n+1} and connect V_n with V_1 in such a way that the new edge (V_n, V_1), together with the two deleted ones, (V_n, V_{n+1}) and (V_{n+1}, V_1), form a contractible triangle. Denote the new polygon by P'.

Let us show that P' is strictly convex. P is strictly convex, therefore through every $d-1$ vertices of P' there passes a hyperplane H intersecting P with multiplicity $d-1$. We want to show that the intersection multiplicity of H with P' is also $d-1$. Let H' be a hyperplane close to H and transverse to P and P'. The intersection number of H' with P' does not exceed that with P. Indeed, if H' intersects the new edge, then it intersects one of the deleted ones since the triangle is contractible.

By the induction hypothesis, P' has at least $d+1$ flattenings. To prove the theorem, it remains for us to show that P' cannot have more flattenings than P.

Consider the sequence of determinants (3.1) $\Delta_1, \Delta_2, \dots, \Delta_{n+1}$. On replacing P by P' we remove $d+1$ consecutive determinants

$$\Delta_{n-d+1},\ \Delta_{n-d+2},\ \dots,\ \Delta_{n+1} \tag{3.4}$$

and replace them with d new determinants

$$\Delta'_{n-d+1},\ \Delta'_{n-d+2},\ \dots,\ \Delta'_{n}, \tag{3.5}$$

where

$$\Delta'_{n-d+i} = \left| \widetilde{V}_{n-d+i} \ \dots\ \widehat{\widetilde{V}}_{n+1} \ \dots\ \widetilde{V}_{n+i+1} \right| \tag{3.6}$$

with $i = 1, \dots, d$. The transition from (3.4) to (3.5) is done in two steps. Firstly, we add (3.5) to (3.4) so that the two sequences alternate, that is, we put Δ'_j between Δ_j and Δ_{j+1}. And secondly, we delete the "old" determinants (3.4). We will prove that the first step preserves the number of sign changes, while the second step obviously cannot increase this number.

LEMMA 3.12. *If Δ_{n-d+i} and $\Delta_{n-d+i+1}$ have the same sign, then Δ'_{n-d+i} is also of the same sign.*

Proof of the lemma. Since P is in general position, the removed vector $\widetilde{V}_{n+1}$ is a linear combination of $d+1$ vectors $\widetilde{V}_{n-d+i}, \dots, \widetilde{V}_n$, $\widetilde{V}_{n+2}, \dots, \widetilde{V}_{n+i+1}$:

$$\widetilde{V}_{n+1} = a\widetilde{V}_{n-d+i} + b\widetilde{V}_{n+i+1} + \cdots, \tag{3.7}$$

where the dots indicate a linear combination of the remaining vectors. It follows from (3.6) that

$$\Delta_{n-d+i} = (-1)^{i-1}\, b\, \Delta'_{n-d+i}, \quad \Delta_{n-d+i+1} = (-1)^{d-i}\, a\, \Delta'_{n-d+i}\,. \tag{3.8}$$

It is time to use the strict convexity of P. Let H be a hyperplane in $\mathbf{RP}^d$ through $d-1$ vertices $V_{n-d+i+1}, \ldots, \widehat{V}_{n+1}, \ldots, V_{n+i}$ which intersects P with multiplicity $d-1$, and let $\widetilde{H}$ be its lifting to $\mathbf{R}^{d+1}$. Choose a linear function φ in $\mathbf{R}^{d+1}$ vanishing on $\widetilde{H}$ and such that $\varphi(\widetilde{V}_{n+1}) > 0$. We claim that

$$(-1)^{d-i}\varphi(\widetilde{V}_{n-d+i}) > 0 \quad \text{and} \quad (-1)^{i-1}\varphi(\widetilde{V}_n) > 0\,. \tag{3.9}$$

Indeed, by Lemma 3.3, the intersection multiplicities of $\widetilde{H}$ with the polygonal lines $(\widetilde{V}_{n-d+i}, \ldots, \widetilde{V}_{n+1})$ and $(\widetilde{V}_{n+1}, \ldots, \widetilde{V}_{n+i+1})$ are at least $d-i$ and $i-1$, respectively. Since H intersects P with multiplicity $d-1$, the above two multiplicities are indeed equal to $d-i$ and $i-1$. The inequalities (3.9) now readily follow from Lemma 3.5.

Finally, we evaluate φ on (3.7):

$$\varphi(\widetilde{V}_{n+1}) = a\,\varphi(\widetilde{V}_{n-d+i}) + b\,\varphi(\widetilde{V}_{n+i+1})\,.$$

It follows from (3.9) and the inequality $\varphi(\widetilde{V}_{n+1}) > 0$ that at least one of the numbers $(-1)^{i-1}b$ and $(-1)^{d-i}a$ is positive. In view of (3.8), Lemma 3.12 follows. □

Thus Theorem 3.11 is also proved. □

REMARK 3.13. Strict convexity is necessary for the existence of $d+1$ flattenings. One can easily construct a closed polygon without any flattenings and even C^0-approximate an arbitrary closed smooth curve by such polygons. In the smooth case such an approximation is well known: given a curve γ_0, the approximating one, γ, spirals around in a tubular neighbourhood of γ_0. In the polygonal case we take a sufficiently fine straightening of γ.

4. APPLICATIONS OF THE MAIN THEOREM

4.1 PROOF OF THEOREMS 2.2, 2.6 AND 2.10

Now we prove the results announced in Section 2. The idea is the same in all three cases and is precisely that of Barner's proof of the smooth versions of these theorems – see [3] and also [15]. We will consider Theorem 2.6 in detail, indicating the necessary changes in the other two cases.