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2. Some classical definitions

We begin with some very general definitions concerning group actions. For an introduction to this subject, we refer to [42].

Let Γ be any group and X be any topological space. An *action* of Γ on X is a homomorphism ϕ from Γ to the group Homeo(X) of homeomorphisms of X. An element $\gamma \in \Gamma$ and a point $x \in X$ produce the point $\gamma \cdot x = \phi(\gamma)(x)$. Conversely a map

$$(\gamma, x) \in \Gamma \times X \mapsto \gamma \cdot x \in X$$

comes from an action if for every γ, the point $\gamma \cdot x$ depends continuously on x and if for every γ_1, γ_2 we have $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1\gamma_2) \cdot x$ and $e.x = x$ (e denotes the identity element in Γ).

Two actions ϕ_1 and ϕ_2 of Γ on X_1 and X_2 are *conjugate* if there exists a homeomorphism h from X_1 to X_2 such that for every $\gamma \in \Gamma$, one has $\phi_2(\gamma) = h\phi_1(\gamma)h^{-1}$.

An action ϕ is *faithful* if it is injective, *i.e.* if non trivial elements in the group act non trivially on the space. This is a minor assumption since we can always consider the associated faithful action of the quotient group $\Gamma/\ker(\phi)$.

The *orbit* of a point x is the set $\mathcal{O}(x) = \{\phi(\gamma)(x) \mid \gamma \in \Gamma\} \subset X$. The main object of topological dynamics is to study the topological properties of the partition of X into orbits. An action is *transitive* if there is only one orbit. We say in this case that X is *homogeneous* under the action of Γ. Of course, these transitive actions are quite trivial from the topological dynamics point of view but this does not mean that the geometrical study of homogeneous spaces is not interesting !

The *stabilizer* of the point x is the subgroup

$$Stab(x) = \{\gamma \in \Gamma \mid \phi(\gamma)(x) = x\} \subset \Gamma .$$

There is a natural bijection between the quotient $\Gamma/Stab(x)$ and the orbit $\mathcal{O}(x)$. Note that the stabilizers of two points in the same orbit are conjugate subgroups in Γ. An action is *free* if the stabilizer of every point is trivial, *i.e.* if the action of a non trivial element of Γ has no fixed point.

In some cases, Γ might be a topological group. In these cases, we frequently consider *continuous actions* such that $\gamma \cdot x$ is a continuous function on $\Gamma \times X$. The orbit map bijection from $\Gamma/Stab(x)$ to $\mathcal{O}(x)$ is continuous but is usually not a homeomorphism when $\mathcal{O}(x)$ is equipped with the induced topology from X. The easiest non trivial example is the case where $\Gamma = \mathbf{R}$, *i.e.* of a topological flow: if the stabilizer of a point x is trivial, the orbit $\mathcal{O}(x)$ is the image of a continuous bijection $\mathbf{R} \to \mathcal{O}(x) \subset X$ but in many cases this orbit might be recurrent (for instance dense in X) and this bijection is not a homeomorphism. There is however a special case in which this map is indeed a homeomorphism and we use this fact constantly (and sometimes implicitely) in these notes. Consider a Lie group G acting continuously and transitively on a manifold M and denote by H the stabilizer of a point. Then H is a closed subgroup of G, hence a closed Lie subgroup, and the quotient space G/H is naturally a smooth manifold. In this case, the orbit map from G/H to M is a homeomorphism.