

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 47 (2001)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GROUPS ACTING ON THE CIRCLE
Autor: GHYS, Étienne
Kapitel: 2. Some classical definitions
DOI: <https://doi.org/10.5169/seals-65441>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 18.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

CONTENTS

1. Introduction	329
2. Some classical definitions	330
3. Two basic examples	332
3.1. The projective group	333
3.2. Piecewise linear groups	336
4. The group of homeomorphisms of the circle	338
4.1. Locally compact groups acting on the circle	345
5. Rotation numbers	349
5.1. Dynamics of a single homomorphism	349
5.2. Tits' alternative	359
6. Bounded Euler class	363
6.1. Group cohomology	363
6.2. The Euler class of a group acting on the circle	366
6.3. Bounded cohomology and the Milnor-Wood inequality	367
6.4. Explicit bounds on the Euler class	372
6.5. Actions of the real line and orderings	373
6.6. Some examples	382
7. Higher rank lattices	386
7.1. Witte's theorem	387
7.2. Actions of higher rank lattices	390
7.3. Lattices in linear groups	392
7.4. Some groups that do act...	401
References	404

2. SOME CLASSICAL DEFINITIONS

We begin with some very general definitions concerning group actions. For an introduction to this subject, we refer to [42].

Let Γ be any group and X be any topological space. An *action* of Γ on X is a homomorphism ϕ from Γ to the group $\text{Homeo}(X)$ of homeomorphisms of X . An element $\gamma \in \Gamma$ and a point $x \in X$ produce the point $\gamma \cdot x = \phi(\gamma)(x)$. Conversely a map

$$(\gamma, x) \in \Gamma \times X \mapsto \gamma \cdot x \in X$$

comes from an action if for every γ , the point $\gamma \cdot x$ depends continuously on x and if for every γ_1, γ_2 we have $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x$ and $e \cdot x = x$ (e denotes the identity element in Γ).

Two actions ϕ_1 and ϕ_2 of Γ on X_1 and X_2 are *conjugate* if there exists a homeomorphism h from X_1 to X_2 such that for every $\gamma \in \Gamma$, one has $\phi_2(\gamma) = h\phi_1(\gamma)h^{-1}$.

An action ϕ is *faithful* if it is injective, *i.e.* if non trivial elements in the group act non trivially on the space. This is a minor assumption since we can always consider the associated faithful action of the quotient group $\Gamma/\ker(\phi)$.

The *orbit* of a point x is the set $\mathcal{O}(x) = \{\phi(\gamma)(x) \mid \gamma \in \Gamma\} \subset X$. The main object of topological dynamics is to study the topological properties of the partition of X into orbits. An action is *transitive* if there is only one orbit. We say in this case that X is *homogeneous* under the action of Γ . Of course, these transitive actions are quite trivial from the topological dynamics point of view but this does not mean that the geometrical study of homogeneous spaces is not interesting!

The *stabilizer* of the point x is the subgroup

$$\text{Stab}(x) = \{\gamma \in \Gamma \mid \phi(\gamma)(x) = x\} \subset \Gamma.$$

There is a natural bijection between the quotient $\Gamma/\text{Stab}(x)$ and the orbit $\mathcal{O}(x)$. Note that the stabilizers of two points in the same orbit are conjugate subgroups in Γ . An action is *free* if the stabilizer of every point is trivial, *i.e.* if the action of a non trivial element of Γ has no fixed point.

In some cases, Γ might be a topological group. In these cases, we frequently consider *continuous actions* such that $\gamma \cdot x$ is a continuous function on $\Gamma \times X$. The orbit map bijection from $\Gamma/\text{Stab}(x)$ to $\mathcal{O}(x)$ is continuous but is usually not a homeomorphism when $\mathcal{O}(x)$ is equipped with the induced topology from X . The easiest non trivial example is the case where $\Gamma = \mathbf{R}$, *i.e.* of a topological flow: if the stabilizer of a point x is trivial, the orbit $\mathcal{O}(x)$ is the image of a continuous bijection $\mathbf{R} \rightarrow \mathcal{O}(x) \subset X$ but in many cases this orbit might be recurrent (for instance dense in X) and this bijection is not a homeomorphism. There is however a special case in which this map is indeed a homeomorphism and we use this fact constantly (and sometimes implicitly) in these notes. Consider a Lie group G acting continuously and transitively on a manifold M and denote by H the stabilizer of a point. Then H is a closed subgroup of G , hence a closed Lie subgroup, and the quotient space G/H is naturally a smooth manifold. In this case, the orbit map from G/H to M is a homeomorphism.