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LATTICES OF COVARIANT QUADRATIC FORMS

by Wilhelm PLESKEN

1. INTRODUCTION

The problem of constructing integral lattices in Euclidean space with big density for the associated sphere packing has attracted considerable attention in the last years; cf. [CoS88]. Some of the lattices found in this context were constructed as G-lattices for some finite group G; cf. [NeP95], [Neb95], [Neb96a], [Neb96b], or [Ple98] for a survey. Other sources of constructions were lattices associated with number fields or semi-simple algebras; cf. [BaM94]. Rather than looking at just one bilinear form on a lattice, the present investigation is geared towards the study of certain families of such forms.

More precisely, a rather general and flexible setting for the $\mathbf{Z}$-lattice $\mathrm{Bil}_{\mathbf{Z}G}(L)$ of all integral G-invariant bilinear forms on a $\mathbf{Z}G$-lattice L is given: one replaces the group ring $\mathbf{Z}G$ by a $\mathbf{Z}$-order Λ with a positive involution and the invariant bilinear forms by covariant ones, as defined in Chapter 2. One learns from [Opg96] and [Opg01] that one should look at the dual lattice L^* at the same time. As pointed out by J.-P. Tignol, the endomorphism ring $\mathrm{End}_\Lambda(L \oplus L^*)$ accommodates both the integral bilinear covariant forms on L and on its dual L^*. Even if the two orders Λ_i with involutions and lattices L_i are completely different, it now becomes natural to consider that the two lattices $\mathrm{Bil}_{\Lambda_1}(L_1)$ and $\mathrm{Bil}_{\Lambda_2}(L_2)$ of bilinear covariant forms on Λ_i-lattices L_i are equivalent if the endomorphism rings $\mathrm{End}_{\Lambda_i}(L_i \oplus L_i^*)$ are isomorphic; cf. Chapter 2 for a more precise definition.

In this way the lattice of all integral bilinear forms on the $\mathbf{Z}$-lattice $\mathbf{Z}^n$ becomes equivalent to $\mathrm{Bil}_{\mathbf{Z}G}(\bigoplus^n M)$ for any absolutely irreducible $\mathbf{Z}G$-lattice M admitting a unimodular G-invariant bilinear form. However, the situation is more interesting if the endomorphism rings $\mathrm{End}_\Lambda(L \oplus L^*)$ are not maximal orders, for instance hereditary, to mention the next simplest case. In Chapter 4 a canonical process is described which associates with each $\mathrm{Bil}_\Lambda(L)$ a Λ-lattice $\bar{L}$ such that $\mathrm{End}_\Lambda(\bar{L} \oplus \bar{L}^*)$ is hereditary. At the same time one gets an invariant called the e-$*$-depth measuring how far away L, resp. $\mathrm{Bil}_\Lambda(L)$, is from this well behaved situation.

This process generalizes Watson's process for constructing elementary quadratic forms out of arbitrary integral quadratic forms; cf. [Wat62] (where 'elementary' means that the exponent of the discriminant group is square free). Indeed, the present investigation can also be viewed as a generalization of the study of a single positive definite integral bilinear form ϕ, at least if ϕ is primitive, i.e. surjective onto $\mathbf{Z}$, namely by obtaining $\mathbf{Z}\phi$ as $\mathrm{Bil}_\Lambda(L)$. Equivalence then means that the exponents of the discriminant groups (= biggest elementary divisors of the Gram matrices) are equal for the two primitive forms considered. It should be noted that the general procedure applied here is called the *radical idealizer process* and is quite common in the general theory of orders.

In Chapter 3 the group of autoequivalences is studied without using the underlying lattice L in any serious way other than via $\mathrm{End}_\Lambda(L \oplus L^*)$. The notions depth and $*$-depth for $\mathrm{Bil}_\Lambda(L)$ measure how far $\mathrm{End}_\Lambda(L)$ and $\mathrm{End}_\Lambda(L \oplus L^*)$ are away from being hereditary. The $*$-depth zero situations are often classifiable. In the depth zero situation structural results on the outer group of autoequivalences can be given. Even more restrictions for the outer automorphism group in the $*$-depth zero case are given in Chapter 4.

Chapter 5 studies the special situation where inversion of nondegenerate forms can be rescaled to become a $\mathbf{Z}$-linear mapping of the nondegenerate elements in $\mathrm{Bil}_\Lambda(L)$ into $\mathrm{Bil}_\Lambda(L^*)$. Quebbemann's definition of modular lattices, cf. [Que95] and [Que97], is taken up to define $\mathrm{Bil}_\Lambda(L)$ to be modular if there is a simultaneous modularity transformation for all positive definite forms in $\mathrm{Bil}_\Lambda(L)$. Finally, in Chapter 6, some examples are studied, e.g. if $\mathrm{End}_\Lambda(L)$ is a $\mathbf{Z}$-order in the algebra $\mathbf{Q}^{2\times 2}$. Examples of this nature have also been studied by Bavard, cf. [Bav97], in a geometric manner in the context of symplectic lattices.

Whenever something new is introduced, one should justify it by giving the benefits for the old problems. So, for instance, the present investigations give a better understanding of the normalizer of a finite unimodular group within the full unimodular group (cf. discussion of $N(L)$ following Definition 4.4). The sort of insight one gets into the structure of the normalizer allows one to compare normalizers in their actions on the $\mathrm{Bil}_\Lambda(L)$ even if the groups are of different degrees.

In this sense the examples at the end of the paper describe infinitely many normalizers. The reader who wants to look at some other, more concrete, examples might use the package [1]) *CARAT®* handling low-dimensional crystallographic groups; cf. [OPS98] or [PS00]. Here are some further applications of the present investigations: they help to check when two finite unimodular groups are conjugate in the full unimodular group by comparing the lattices of invariant forms; they help to create models of such lattices in low dimensions by passing to equivalent lattices of covariant forms; they help to find candidates for lattices of covariant forms which contain interesting positive definite bilinear forms, and to locate these forms inside the lattice of covariant forms.

It is a pleasure to acknowledge many inspiring discussions with G. Nebe.

2. Covariant forms and equivalence

Throughout the paper, $\mathcal{A}$ denotes a semi-simple $\mathbf{Q}$-algebra with a positive involution $^\circ$, i.e. an antiautomorphism of order two of $\mathcal{A}$ such that $\mathcal{A} \to \mathbf{Q}$: $a \mapsto tr_{\mathcal{A}/\mathbf{Q}}(aa^\circ)$ is a positive definite quadratic form on $\mathcal{A}$, where $tr_{\mathcal{A}/\mathbf{Q}}$ denotes the reduced trace of $\mathcal{A}$. Together with $\mathcal{A}$, fix a faithful finite dimensional right $\mathcal{A}$-module $\mathcal{V}$. The basic data to start with are $\mathcal{A}$, $^\circ$, and L, where L is a full $\mathbf{Z}$-lattice in $\mathcal{V} = L_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} L$. Because of the involution, $\mathcal{V}^* := \mathrm{Hom}_{\mathbf{Q}}(\mathcal{V}, \mathbf{Q})$ becomes a right $\mathcal{A}$-module again, which is isomorphic to $\mathcal{V}$. Inside $\mathcal{V}^*$ one has $L^* := \{\varphi \in \mathcal{V}^* \mid L\varphi \subset \mathbf{Z}\}$, which can be identified with $\mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$.

Definition 2.1.

(i) $\Lambda(L) := \{a \in \mathcal{A} \mid La \subseteq L \text{ and } L^*a \subseteq L^*\}$ is called the $^\circ$*-invariant* order of L in $\mathcal{A}$.

(ii) A $\mathbf{Z}$-bilinear form $\phi \colon L \times L \to \mathbf{Z}$ is called *covariant* (with respect to $^\circ$) if it satisfies

$$\phi(Va, W) = \phi(V, Wa^\circ) \text{ for all } V, W \in L,\ a \in \Lambda\,,$$

where Λ is any $^\circ$-invariant $\mathbf{Z}$-order in $\mathcal{A}$, contained in $\Lambda(L)$ of finite index.

(iii) The $\mathbf{Z}$-lattice of all, resp. all symmetric or skew-symmetric, covariant $\mathbf{Z}$-bilinear forms on L is denoted by $\mathrm{Bil}_\Lambda(L)$, resp. $\mathrm{Bil}_\Lambda^+(L)$ or $\mathrm{Bil}_\Lambda^-(L)$. Finally $\mathrm{Bil}_{\Lambda,>0}^+(L)$ denotes the set of positive definite elements in $\mathrm{Bil}_\Lambda^+(L)$.

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[1]) This is available via internet http://wwwb.math.rwth-aachen.de/carat/index.html.

Extending this notation for any commutative ring R containing $\mathbf{Z}$, one can consider covariant R-valued bilinear forms. They give rise to the R-modules $\mathrm{Bil}_{\Lambda_R}(L_R)$, resp. $\mathrm{Bil}^+_{\Lambda_R}(L_R)$ and $\mathrm{Bil}^-_{\Lambda_R}(L_R)$, spanned by the above $\mathbf{Z}$-lattices. If R is contained in $\mathbf{R}$, $\mathrm{Bil}^+_{\Lambda_R,>0}(L_R)$ denotes the set of positive definite elements in $\mathrm{Bil}^+_{\Lambda_R}(L_R)$. One checks, that $\mathrm{Bil}^+_{\Lambda_\mathbf{R},>0}(L_\mathbf{R})$ is an open, nonempty cone in the real vector space $\mathrm{Bil}^+_{\Lambda_\mathbf{R}}(L_\mathbf{R})$. Any nondegenerate element of $\mathrm{Bil}_{\mathcal{A}}(L_\mathbf{Q})$ can be used to recover the involution $^\circ$ on $\mathcal{A}$. To connect covariance with the more familiar notion of a sesquilinear form – cf. [Scha85], p. 236, [BaF96] –, one should note that composition with the reduced trace of $\mathcal{A}$ yields a $\mathbf{Z}$-isomorphism of the lattice of sesquilinear maps of L taking values in the inverse different of $\Lambda(L)$ onto $\mathrm{Bil}_\Lambda(L)$. Three typical examples will demonstrate the generality of the concept:

EXAMPLE 2.2.

(i) Fix a positive definite symmetric matrix $f \in \mathbf{Q}^{n\times n}$. Let $\mathcal{A} = \mathbf{Q}^{n\times n}$ with $a^\circ = fa^{tr}f^{-1}$ for all $a \in \mathcal{A}$ and let $L = \mathbf{Z}^{1\times n}$. There is a unique positive definite rational multiple f_0 of f, which is *integral* and *primitive*, i. e. the greatest common divisor of the entries of f_0 is 1. One checks that $\mathrm{Bil}_\Lambda(L) = \mathbf{Z}f_0$ and $\mathrm{Bil}^+_{\Lambda,>0}(L) = \mathbf{N}f_0$. If f_0 is unimodular, then $\Lambda(L) = \mathbf{Z}^{n\times n}$, any other $\Lambda(L)$-lattice is of the form $\bigoplus^k L$, and $\mathrm{Bil}_\Lambda(\bigoplus^k L) = \{x \otimes f_0 \mid x \in \mathbf{Z}^{k\times k}\}$, where $\otimes$ denotes the Kronecker product (of two matrices). Note that $\mathrm{End}_{\Lambda(L)}(\bigoplus^k L) \cong \mathbf{Z}^{k\times k}$.

(ii) Let $G \leq \mathrm{GL}_n(\mathbf{Z})$ be a finite unimodular group. Set $\mathcal{A} := \overline{\mathbf{Q}G}$ the enveloping algebra of G, i. e. the subalgebra of $\mathbf{Q}^{n\times n}$ spanned by the matrices of G (clearly an epimorphic image of the group algebra $\mathbf{Q}G$) and let $L := \mathbf{Z}^{1\times n}$. Obviously the standard involution $g \mapsto g^{-1}$ for $g \in G$ of $\mathbf{Q}G$ induces a positive involution on $\overline{\mathbf{Q}G}$. The order $\Lambda(L)$ contains $\overline{\mathbf{Z}G}$, the $\mathbf{Z}$-span of the matrices of G as a suborder of finite index. $\mathrm{Bil}_\Lambda(L)$ consists of all G-invariant bilinear forms. $\mathrm{Bil}^+_{\Lambda_\mathbf{R},>0}(L_\mathbf{R})$ is known as the *Bravais manifold* of G. If there is no finite unimodular group H containing G properly with the same $\mathrm{Bil}_\Lambda(L)$, resp. $\mathrm{Bil}^+_\Lambda(L)$, then G is called the *strict Bravais group*, resp. the *Bravais group*; cf. [OPS98].

(iii) Up to isomorphism there are three types of real simple algebras with a definite involution, namely $(\mathbf{R}^{n\times n}, {}^{tr})$, $(\mathbf{C}^{n\times n}, {}^{-tr})$, and $(\mathbf{H}^{n\times n}, {}^{-tr})$, where $^-$ denotes complex, resp. quaternionic, conjugation. A (right) module for such $K^{n\times n}$ can be taken to be $K^{s\times n}$ with endomorphism ring $K^{s\times s}$ according to the three possibilities for K above. Then the $\mathbf{R}$-space of covariant forms can also be represented by $K^{s\times s}$, where the symmetric forms correspond to the symmetric matrices in case $K = \mathbf{R}$ and to the Hermitian matrices in the remaining two cases. According to the decomposition of $\mathcal{A}_{\mathbf{R}}$ into such simple components, one clearly has a decomposition of $\mathrm{Bil}_{\mathcal{A}_{\mathbf{R}}}(L_{\mathbf{R}})$ into components, each of which can be described as such a $K^{s\times s}$ with suitable K and s as above. In particular, this gives the $\mathbf{Z}$-ranks of $\mathrm{Bil}_{\Lambda}(L)$, $\mathrm{Bil}_{\Lambda}^{+}(L)$, and $\mathrm{Bil}_{\Lambda}^{-}(L)$.

It is often helpful to identify $\mathrm{Bil}_{\Lambda}(L) = \mathrm{Bil}_{\Lambda(L)}(L)$ with $\mathrm{Hom}_{\Lambda}(L, L^*) = \mathrm{Hom}_{\Lambda(L)}(L, L^*)$ as $\mathbf{Z}$-lattices. More precisely $\phi \in \mathrm{Bil}_{\Lambda}(L)$ is identified with $\tau \in \mathrm{Hom}_{\Lambda(L)}(L, L^*)$ by $W(\tau(V)) := \phi(V, W)$ for all $V, W \in L$, where we write $\Lambda(L)$-homomorphisms of right $\Lambda(L)$-modules on the left. As $\Lambda(L)$ can be replaced by any suborder Λ of $\Lambda(L)$ of finite index (invariant under the involution), we shall usually write Λ instead of $\Lambda(L)$ in the sequel. In this way, $\mathrm{Bil}_{\Lambda}(L^*)$ is also identified with $\mathrm{Hom}_{\Lambda}(L^*, L)$ and one gets bilinear maps $\mathrm{Bil}_{\Lambda}(L) \times \mathrm{Bil}_{\Lambda}(L^*) \to \mathrm{End}_{\Lambda}(L^*)$ and $\mathrm{Bil}_{\Lambda}(L^*) \times \mathrm{Bil}_{\Lambda}(L) \to \mathrm{End}_{\Lambda}(L)$, which can be composed with the reduced traces of the endomorphism rings of $L^*_{\mathbf{Q}}$ and of $L_{\mathbf{Q}}$ respectively, to obtain $\mathbf{Z}$-valued bilinear maps. Of course the latter become nondegenerate pairings if one tensors with the field of rational numbers. Hence one gets a discriminant for $\mathrm{Bil}_{\Lambda}(L)$, which measures the deviation of $(\mathrm{Bil}_{\Lambda}(L), \mathrm{Bil}_{\Lambda}(L^*))$ from being in perfect duality. Obviously, the same can be done for $\mathrm{Bil}_{\Lambda}^{+}(L)$ and $\mathrm{Bil}_{\Lambda}^{-}(L)$.

DEFINITION 2.3. Let ϵ stand for the empty symbol, $+$, or $-$. The *discriminant* of the pair $(\mathrm{Bil}_{\Lambda}^{\epsilon}(L), \mathrm{Bil}_{\Lambda}^{\epsilon}(L^*))$ is defined as

$$\mathrm{discr}\left(\mathrm{Bil}_{\Lambda}^{\epsilon}(L), \mathrm{Bil}_{\Lambda}^{\epsilon}(L^*)\right) := \left|\det\left(\mathrm{Tr}(\phi_i\psi_j)\right)_{1\le i,j\le d}\right|,$$

where $(\phi_1, \ldots, \phi_d)$, resp. $(\psi_1, \ldots, \psi_d)$, form $\mathbf{Z}$-bases of $\mathrm{Bil}_{\Lambda}^{\epsilon}(L)$, resp. $\mathrm{Bil}_{\Lambda}^{\epsilon}(L^*)$, and Tr denotes the reduced trace of $\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$.

Clearly, the definitions are independent of the choice of bases and one can even define a discriminant group, whose order is the discriminant. As an easy exercise the reader may check that in the case of Example 2.2 (i) the discriminant $\mathrm{discr}(\mathrm{Bil}_{\Lambda}^{+}(L)), \mathrm{Bil}_{\Lambda}^{+}(L^*))$ is equal to the exponent of the discriminant group $L^{\sharp,f_0}/L$ of (L, ϕ_0), where $L^{\sharp,f_0} := \{V \in \mathcal{V} \mid \phi_0(L, V) \subseteq \mathbf{Z}\}$ with ϕ_0 the bilinear form described by f_0.

Another observation along the lines of the interplay between $\mathrm{Bil}_{\Lambda}(L)$, $\mathrm{Bil}_{\Lambda}(L^*)$, $\mathrm{End}_{\Lambda}(L)$, and $\mathrm{End}_{\Lambda}(L^*)$ is the presence of all of these in $\mathrm{End}_{\Lambda}(L \oplus L^*)$.

REMARK 2.4.

$$\mathrm{End}_{\mathcal{A}}(\mathcal{V}\oplus\mathcal{V}^*) = \begin{pmatrix} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) & \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*,\mathcal{V}) \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*) & \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*) \end{pmatrix}$$

is a $\mathbf{Q}$-algebra with involution $\begin{pmatrix} \zeta & \psi \\ \phi & \eta \end{pmatrix} \mapsto \begin{pmatrix} \eta^{tr} & \psi^{tr} \\ \phi^{tr} & \zeta^{tr} \end{pmatrix}$ and a C_2-graduation

$$\begin{pmatrix} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) & 0 \\ 0 & \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*) \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*,\mathcal{V}) \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*) & 0 \end{pmatrix}.$$

The involution is induced by the symmetric bilinear form ν on $\mathcal{V}\oplus\mathcal{V}^*$ defined by

$$\nu\colon (\mathcal{V}\oplus\mathcal{V}^*)\times(\mathcal{V}\oplus\mathcal{V}^*) \to \mathbf{Q} : \big((V_1,\varphi_1),(V_2,\varphi_2)\big) \mapsto V_1\varphi_2 + V_2\varphi_1$$

With respect to this bilinear form one has $(X\oplus Y)^\# = Y^*\oplus X^*$ for any two full lattices $X\subset\mathcal{V}$ and $Y\subset\mathcal{V}^*$. In particular, $\mathrm{End}_\Lambda(L\oplus L^*)$ is invariant under the involution.

The following proposition yields a better understanding of the discriminant.

PROPOSITION 2.5.

$$\mathrm{discr}\left(\mathrm{Bil}_\Lambda(L),\mathrm{Bil}_\Lambda(L^*)\right)\cdot \mathrm{discr}\left(\mathrm{End}_\Lambda(L)\right)^2 = \left|\mathrm{discr}\left(\mathrm{End}_\Lambda(L\oplus L^*)\right)\right|,$$

where the discriminants are taken with respect to the reduced traces.

Proof. One has $\mathrm{End}_\Lambda(L\oplus L^*) =$

$$\begin{pmatrix} \mathrm{End}_\Lambda(L) & 0 \\ 0 & \mathrm{End}_\Lambda(L^*) \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathrm{Hom}_\Lambda(L^*,L) \\ \mathrm{Hom}_\Lambda(L,L^*) & 0 \end{pmatrix}.$$

Since the two summands are orthogonal to each other with respect to the trace bilinear form, and since $\mathrm{End}_\Lambda(L)$ and $\mathrm{End}_\Lambda(L^*)$ are antiisomorphic and therefore have the same discriminant, the claim follows. □

Now the basic definition is well motivated.

DEFINITION 2.6. Let $(\mathcal{B},{}^\circ)$ be a $\mathbf{Q}$-algebra with a positive involution, and $\mathcal{W}$ a (faithful) $\mathcal{B}$-module containing a full $\mathbf{Z}$-lattice M. Let Γ be some suborder of finite index in $\Lambda(M)$. Finally let R be some subring of $\mathbf{R}$ containing $\mathbf{Z}$. We say that $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Gamma(M)$ are *R-equivalent* if there exists an R-module isomorphism $\omega\colon \mathrm{Bil}_{\Lambda_R}(L_R)\to\mathrm{Bil}_{\Lambda_R}(M_R)$, called an *$R$-equivalence*, which extends to an isomorphism Ω of R-algebras with involution and C_2-grading from $\mathrm{End}_{\Lambda_R}((L\oplus L^*)_R)$ onto $\mathrm{End}_{\Gamma_R}((M\oplus M^*)_R)$ and which induces a bijection from $\mathrm{Bil}^+_{\Lambda_R,>0}(L_R)$ onto $\mathrm{Bil}^+_{\Gamma_R,>0}(M_R)$. If $R=\mathbf{Z}$ then one simply says *equivalence* instead of $\mathbf{Z}$-equivalence.

It is worthwhile to spell out the isomorphism of $\mathrm{End}_{\Lambda_R}((L \oplus L^*)_R)$ onto $\mathrm{End}_{\Gamma_R}((M \oplus M^*)_R)$ in more detail. The equivalence ω: $\mathrm{Bil}_\Lambda(L_R) \to \mathrm{Bil}_{\Gamma_R}(M_R)$ obviously induces an R-module isomorphism ω': $\mathrm{Bil}_{\Lambda_R}(L_R^*) \to \mathrm{Bil}_{\Gamma_R}(M_R^*)$, for one may assume $R = \mathbf{R}$ and $\mathrm{Bil}_\Lambda(L_\mathbf{R})$ is spanned by nondegenerate (resp. invertible) elements ψ, and accordingly $\mathrm{Bil}_{\Lambda_\mathbf{R}}(L_\mathbf{R}^*)$ by the ψ^{-1}. The relation $\psi^{-1}\psi = id_{L_\mathbf{R}}$ translates into $\omega'(\psi^{-1}) = (\omega(\psi))^{-1}$. Obviously ω and ω', taken together, yield unique R-algebra isomorphisms ω_1: $\mathrm{End}_{\Lambda_R}(L_R) \to \mathrm{End}_{\Gamma_R}(M_R)$ and ω_2: $\mathrm{End}_{\Lambda_R}(L_R^*) \to \mathrm{End}_{\Gamma_R}((M_R^*)$, which are related by $\omega_2(\eta) = (\omega_1(\eta^{tr}))^{tr}$ for all $\eta \in \mathrm{End}_{\Gamma_R}((L_R^*)$. So one has the following

REMARK 2.7. In Definition 2.6 the R-algebra isomorphism

$$\Omega\colon \mathrm{End}_{\Lambda_R}((L \oplus L^*)_R) \to \mathrm{End}_{\Gamma_R}((M \oplus M^*)_R)$$

is uniquely determined by the equivalence ω: $\mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(M_R)$.

Obviously the discriminant of the pair $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*))$ and the discriminant group of $\mathrm{Bil}_\Lambda(L)$ do not change when one passes to an equivalent lattice of covariant forms. In the case of one-dimensional spaces of compatible forms, the discriminant separates equivalence classes.

REMARK 2.8. In the situation of Definition 2.6 let $\mathrm{Bil}_\mathcal{A}(\mathcal{V})$ and $\mathrm{Bil}_\mathcal{B}(\mathcal{W})$ be both one-dimensional. Then $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Lambda(M)$ are equivalent if and only if

$$\mathrm{discr}\,(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*)) = \mathrm{discr}\,(\mathrm{Bil}_\Gamma(M), \mathrm{Bil}_\Gamma(M^*))\,.$$

Proof. The missing direction follows from the following description of $\mathrm{End}_\Lambda(L \oplus L^*)$. Let $d := \mathrm{discr}\,(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*))$ and $\mathrm{Bil}_\Lambda(L) = \mathbf{Z}\phi$. Then

$$\mathrm{End}_\Lambda(L \oplus L^*) = \begin{pmatrix} \mathbf{Z}id_L & \mathbf{Z}d\phi^{-1} \\ \mathbf{Z}\phi & \mathbf{Z}id_{L^*} \end{pmatrix} \cong \begin{pmatrix} \mathbf{Z} & \mathbf{Z}d \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}. \qquad \square$$

From the discussion in Example 2.2 and the definition, it is reasonably clear that $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Lambda(M)$ are $\mathbf{R}$-equivalent if and only if $\mathrm{End}_{\mathcal{A}_\mathbf{R}}(\mathcal{V}_\mathbf{R})$ and $\mathrm{End}_{\mathcal{B}_\mathbf{R}}(\mathcal{W}_\mathbf{R})$ are isomorphic. For $\mathbf{Q}$-equivalence the statement is more difficult to prove.

PROPOSITION 2.9. *Let $(\mathcal{B}, {}^\circ)$ be a $\mathbf{Q}$-algebra with a positive involution, $\mathcal{W}$ a faithful $\mathcal{B}$-module containing a full $\mathbf{Z}$-lattice M, and let $\Gamma := \Lambda(M)$. Then two lattices $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Gamma(M)$ of covariant forms are $\mathbf{Q}$-equivalent if and only if $\mathrm{End}_\mathcal{A}(\mathcal{V})$ and $\mathrm{End}_\mathcal{B}(\mathcal{W})$ are isomorphic as $\mathbf{Q}$-algebras.*

Proof. By Definition 2.6 equivalence of $\mathcal{V}$ and $\mathcal{W}$ implies that the endomorphism rings are isomorphic. To prove the converse, we may assume without loss of generality that $\mathcal{A}$ and $\mathcal{B}$ are simple. Then the endomorphism rings are also simple. Fix an isomorphism $\omega_1\colon \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \to \mathrm{End}_{\mathcal{B}}(\mathcal{W})$. Then $\omega_2\colon \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*) \to \mathrm{End}_{\mathcal{B}}(\mathcal{W}^*) : \eta \mapsto (\omega_1(\eta^{tr}))^{tr}$ is also an isomorphism and $(\omega_2(\zeta^{tr}))^{tr} = \omega_1(\zeta)$ for all $\zeta \in \mathrm{End}_{\mathcal{A}}(\mathcal{V})$. To shorten the notation, set $\mathcal{E} := \mathrm{End}_{\mathcal{A}}(\mathcal{V})$ and $\mathcal{E}' := \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$. (Note, transposing induces an antiisomorphism between $\mathcal{E}$ and $\mathcal{E}'$.)

The next aim is to find a suitable map

$$\omega\colon \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{W},\mathcal{W}^*)$$

as required in Definition 2.6. Clearly, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*)$ is a simple $(\mathcal{E}',\mathcal{E})$-bimodule. The two isomorphisms ω_1 and ω_2 can be used to turn $\mathrm{Hom}_{\mathcal{B}}(\mathcal{W},\mathcal{W}^*)$ into a simple $(\mathcal{E}',\mathcal{E})$-bimodule as well. Then ω lies in

$$H := \mathrm{Hom}_{(\mathcal{E}',\mathcal{E})}(\mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*), \mathrm{Hom}_{\mathcal{B}}(\mathcal{W},\mathcal{W}^*))\,,$$

which is a one-dimensional Z-module, where Z is the centre of $\mathcal{E}$. To get the right identification of Z with the centre of $\mathcal{E}'$, note that the centres of $\mathcal{A}$ and $\mathcal{E}$ can be identified via their action on $\mathcal{V}$ and that $z \mapsto (z^\circ)^{tr}$ therefore gives the right identification of Z with $Z(\mathcal{E}')$.

Now some properties of H have to be investigated: For each $h \in H$ define h^{tr} by $h^{tr}(\phi) := h(\phi^{tr})^{tr}$ for all $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*,\mathcal{V})$. This defines a Z-semilinear action of the cyclic group of order 2 on H. Indeed, one easily checks: $(h^{tr})^{tr} = h$ and $(zh)^{tr} = z^\circ h^{tr}$ for all $z \in Z$ and all $h \in H$. Next, one verifies that there exists a nonzero $h \in H$ with $h^{tr} = h$. Indeed, if $^\circ$ fixes Z pointwise, any $h \in H$ is fixed by ^{tr}, because the subspaces of symmetric and skewsymmetric forms have different dimensions in this case. If $^\circ$ does not fix Z pointwise, the existence of an $h \in H$ with $h^{tr} = h$ follows from a straightforward analysis of semilinear C_2-actions. In any case, the $h \in H$ with $h^{tr} = h$ form a one-dimensional $\widetilde{Z}$-subspace $\widetilde{H}$ of H, where $\widetilde{Z}$ is the $^\circ$-fixed subfield of Z. It is clear that any symmetric $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*) = \mathrm{Bil}_\Lambda(\mathcal{V})$ is mapped onto a symmetric $\omega(\phi)$ by any $\omega \in \widetilde{H}$. The final point is that ω can be chosen in such a way that positive definite forms map onto positive definite ones. This can easily be seen for the ground field $\mathbf{R}$ by the classification of the simple $\mathbf{R}$-algebras with positive involutions. The present case of rational ground field can be reduced to the previous case, i.e. if $h \in \widetilde{H}$ does not respect positive definite forms, then there exists a $z \in \widetilde{Z}$ with the right sign combinations in the various archimedean completions of $\widetilde{Z}$ such that zh maps positive forms onto positive ones. One ends up with a nonzero

$\omega \in \widetilde{H}$ respecting positiveness, which is unique up to multiplication with totally positive elements in $\widetilde{Z}$.

Similarly one finds a suitable map ω': $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*, \mathcal{V}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{W}^*, \mathcal{W})$ as required in Definition 2.6. Finally, to make ω' unique, one requires $\omega'(\phi^{-1}) = \omega(\phi)^{-1}$ for one (and hence for all) invertible $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$. Now it is a routine matter to check that $(\omega_1, \omega_2, \omega, \omega')$ defines an algebra isomorphism Ω of $\mathrm{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*)$ onto $\mathrm{End}_{\mathcal{B}}(\mathcal{W} \oplus \mathcal{W}^*)$ with the required properties. $\square$

At the end of this basic chapter some comments might be in place: The reader should check as a little exercise that $\mathrm{Bil}_\Lambda(L)$ (given as explicit bilinear forms or as maps from L to L^*) determines $\mathrm{End}_\Lambda(L)$ (but not conversely of course) and $\mathrm{End}_\Lambda(L \oplus L^*)$. One now may ask how much is determined by $\mathrm{Bil}_\Lambda^+(L)$.

DEFINITION 2.10. Call L, $\mathcal{V}$ and $\mathrm{Bil}_\Lambda(L)$ *exceptional*, if $\mathrm{End}_{\mathcal{A}_{\mathbf{R}}}(\mathcal{V}_{\mathbf{R}})$ has a simple component isomorphic to $\mathbf{C}$ or $\mathbf{H}$.

REMARK 2.11. The following three conditions are equivalent.

(i) $\mathrm{Bil}_\Lambda^-(L)$ can be recovered from $\mathrm{Bil}_\Lambda^+(L)$;

(ii) $\mathrm{End}_\Lambda(L)$ can be recovered from $\mathrm{Bil}_\Lambda^+(L)$;

(iii) L is not exceptional.

For instance the difference between the Bravais group and the strict Bravais group in Example 2.2 (ii) only occurs in the exceptional situation.

3. AUTOEQUIVALENCES AND INVARIANTS

The basic notation is kept: $(\mathcal{A}, {}^\circ)$, $L \subset \mathcal{V}$, $\mathrm{Bil}_\Lambda(L) \equiv \mathrm{Hom}_\Lambda(L, L^*)$. Continuing Definition 2.6 in the direction 'autoequivalences', we fix the following notation.

DEFINITION 3.1. Let R be a subring of $\mathbf{R}$ containing $\mathbf{Z}$. The group of all R-equivalences ω: $\mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_\Lambda(L_R)$ is denoted by $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))$.

From the discussion following Definition 2.6, it is clear that one has a monomorphism of $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L_R))$ into the group of all automorphisms of $\mathrm{End}_{\Lambda_R}((L \oplus L^*)_R)$ as a C_2-graded algebra with involution, and also into the automorphism group of $\mathrm{End}_{\Lambda_R}(L_R)$. It therefore makes sense to look at the pointwise stabilizer of the centre of $\mathrm{End}_{\Lambda_R}(L_R)$.

REMARK 3.2. Denote by $\mathrm{Aut}_z^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ the biggest subgroup of $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L_R))$ fixing the centre of $\mathrm{End}_{\Lambda_R}(L_R)$

(i) $\mathrm{Aut}_z^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ is a normal subgroup of $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ of finite index with the factor group $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))/\,\mathrm{Aut}_z^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ acting faithfully on the centre of $\mathrm{End}_{\Lambda_R}(L_R)$.

(ii) $\mathrm{Aut}_z^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ is isomorphic to the group of inner automorphisms of $\mathrm{End}_{\Lambda_R}(L_R)$ in case R is a field.

(iii) If R is not a field, let Q be its field of fractions. Then $\mathrm{Aut}_z^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ is isomorphic to a subgroup of $\mathrm{Aut}_z^e(\mathrm{Bil}_{\Lambda_Q}(L_Q))$.

Proof. (i) Finite dimensional semisimple commutative algebras have finite automorphism groups. The same applies to R-orders in such algebras.

(ii) This follows from the Skolem-Noether Theorem.

(iii) Obvious. □

PROPOSITION 3.3. *The group* $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ *acts properly discontinuously on* $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(L_{\mathbf{R}})$.

Proof. That $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ acts on $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(L_{\mathbf{R}})$ follows from the definition of equivalence. By Remark 3.2 it suffices to show that $\mathrm{Aut}_z^e(\mathrm{Bil}_\Lambda(L))$ acts properly discontinuously. But this follows from the well known fact that $\mathrm{GL}_n(\mathbf{Z})$ acts properly discontinuously on the cone of positive definite symmetric matrices of degree n. □

In fact, the action is even discontinuous on $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(L_{\mathbf{R}})$ modulo the action of $\mathbf{R}_{>0}$ by multiplication and, apart from some marginal exceptions, it is also faithful. One interesting issue is the structure and size of $\mathrm{Out}_z^e(\mathrm{Bil}_\Lambda(L))$, to be defined now.

DEFINITION 3.4.

(i) The subgroup of $\mathrm{Aut}_z^e(\mathrm{Bil}_\Lambda(L))$ corresponding to the inner automorphisms of $\mathrm{End}_\Lambda(L)$ will be denoted by $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$ and referred to as *the group of inner automorphisms of* $\mathrm{Bil}_\Lambda(L)$. (Clearly $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L)) \cong \mathrm{Inn}(\mathrm{End}_\Lambda(L)) \cong (\mathrm{End}_\Lambda(L))^*/Z(\mathrm{End}_\Lambda(L)^*)$.)

(ii) Similarly,

$$\mathrm{Out}^e_z(\mathrm{Bil}_\Lambda(L)) := \mathrm{Aut}^e_z(\mathrm{Bil}_\Lambda(L))/\,\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$$

will be *the outer central group of equivalences of* $\mathrm{Bil}_\Lambda(L)$, and

$$\mathrm{Out}^e(\mathrm{Bil}_\Lambda(L)) := \mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))/\,\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$$

the outer group of equivalences of $\mathrm{Bil}_\Lambda(L)$.

$$\mathrm{Out}^e(\mathrm{Bil}(L)) \left\{ \begin{array}{l} \bullet\ \mathrm{Aut}^e(\mathrm{Bil}(L)) \\ \left.\begin{array}{l} \bullet\ \mathrm{Aut}^e_z(\mathrm{Bil}(L)) \\ \bullet\ \mathrm{Inn}(\mathrm{Bil}(L)) \end{array}\right\} \mathrm{Out}^e_z(\mathrm{Bil}(L)) \end{array}\right.$$

PROPOSITION 3.5. *The group* $\mathrm{Out}^e(\mathrm{Bil}_\Lambda(L))$ *is well defined and embeds into* $\mathrm{Out}(\mathrm{End}_\Lambda(L)) := \mathrm{Aut}(\mathrm{End}_\Lambda(L))/\,\mathrm{Inn}(\mathrm{End}_\Lambda(L))$. *In particular, it is finite.*

Proof. Clearly, conjugation by $\omega \in \mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ of an inner automorphism induced by some $\varphi \in \mathrm{End}_\Lambda(L)^*$ results in the inner automorphism induced by $\omega_1(\varphi)$ in the notation of the discussion of Definition 2.6. Hence $\mathrm{Out}^e(\mathrm{Bil}_\Lambda(L))$ is well defined. The finiteness follows from the Jordan-Zassenhaus Theorem, which implies that $\mathrm{Out}(\Gamma)$ is finite for any $\mathbf{Z}$-order Γ in a semisimple $\mathbf{Q}$-algebra, cf. [CuR87] (55.19). □

Obviously $\mathrm{Out}^e_z(\mathrm{Bil}_\Lambda(L))$ is an interesting invariant for the equivalence class of $\mathrm{Bil}_\Lambda(L)$. Further on in this chapter, it will be proved that it is an Abelian group in case $\mathrm{End}_\Lambda(L)$ is hereditary. But some notions from the theory of orders first have to be recalled, in order to define some invariants measuring the distance from this favourable situation.

Recall from [BeZ85] that the *arithmetical radical* $\mathrm{arad}(\Gamma)$ of a $\mathbf{Z}$-order Γ in a semisimple $\mathbf{Q}$-algebra $\mathcal{B}$ is defined as the ideal which localizes to the radical of Γ_p at the primes dividing the discriminant of Γ, and to the localization Γ_p of the order itself at the other primes. The *left idealizer* or *left order* $\Gamma^{(l)}$ of the arithmetical radical $\mathrm{arad}(\Gamma)$ is the biggest $\mathbf{Z}$-order in $\mathcal{B}$ in which $\mathrm{arad}(\Gamma)$ is a left ideal, in particular $\Gamma^{(l)}\,\mathrm{arad}(\Gamma) \subseteq \mathrm{arad}(\Gamma)$. It is well known, cf. [Rei75], that Γ is hereditary if and only if $\Gamma = \Gamma^{(l)}$. Likewise the two-sided idealizer of $\mathrm{arad}(\Gamma)$ is the biggest $\mathbf{Z}$-order in $\mathcal{B}$ having $\mathrm{arad}(\Gamma)$ as a two-sided ideal. It is denoted by $\Gamma^{(rl)}$. A slight modification of the argument in [Rei75] characterizing hereditary orders by the property $\Gamma = \Gamma^{(l)}$ also shows that Γ is hereditary if and only if $\Gamma = \Gamma^{(rl)}$. Besides, if Γ is invariant under an involution of $\mathcal{B}$, so is $\Gamma^{(rl)}$. Define the left, respectively two-sided, *idealizer* *sequence* of Γ by $\Gamma_0 := \Gamma$ and $\Gamma_{i+1} := \Gamma_i^{(l)}$, resp. $\Gamma_{i+1} := \Gamma_i^{(rl)}$, for $i \geq 0$. The *length* of either of these sequences is the smallest i with $\Gamma_i = \Gamma_{i+1}$.

DEFINITION 3.6.

(i) The *e-depth* of $\mathrm{Bil}_\Lambda(L)$ is defined as the length of the left idealizer sequence of $\mathrm{End}_\Lambda(L)$.

(ii) The *e-$*$-depth* of $\mathrm{Bil}_\Lambda(L)$ is defined as the length of the two-sided idealizer sequence of $\mathrm{End}_\Lambda(L \oplus L^*)$.

Clearly, e-depth and e-$*$-depth are well defined and compatible with equivalence. As for the definition of e-$*$-depth, note that all members of the two-sided idealizer sequence of $\mathrm{End}_\Lambda(L \oplus L^*)$ are both C_2-graded and invariant under the involution of $\mathrm{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*)$. However, it does not seem that they are necessarily endomorphism rings of lattices $M \oplus M^*$ with the M's constructed from L in a canonical way. That is why we shall focus here mainly on the e-$*$-depth, resp. e-depth, zero case. The general discussion will be resumed in the next section; cf. 4.8 and 4.10. Already the case of one-dimensional $\mathrm{Bil}_\Lambda(L)$ shows that even if the e-depth is zero, the e-$*$-depth can be arbitrarily large, since the discriminant of $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}(L^*))$ can be arbitrarily big. However, it seems that for every isomorphism type of $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ the equivalence classes of e-$*$-depth 0 lattices $\mathrm{Bil}_\Lambda(L)$ of covariant integral forms can be classified, provided one restricts the number of primes involved in the discriminant. Here is an example, whose verification is left to the reader as an exercise in combinatorics.

EXAMPLE 3.7. Let $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$. Assume that $\mathrm{Bil}_\Lambda(L)$ is of e-$*$-depth zero and that the discriminant of the pair $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}(L^*))$ is a power of a prime p. Then there are nine equivalence classes of such lattices and the endomorphism rings satisfy $\mathrm{End}_\Lambda(L \oplus L^*) \cong X(E)$ with E one of the matrices

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&1&1\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&1&1\\0&0&0&1\\0&-1&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&1\\1&0&1&1\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\1&0&1&1\\0&0&0&1\\0&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\1&0&1&2\\0&0&0&1\\0&-1&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&1\\1&0&1&2\\0&-1&0&1\\-1&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&2\\1&0&2&2\\0&-1&0&1\\-1&-1&0&0 \end{pmatrix}$$

and $X((m_{ij})) := \{(x_{ij}) \in \mathbf{Q}^{n\times n} \mid x_{ij} \in p^{m_{ij}}\mathbf{Z}\}$ for any $(m_{ij}) \in \mathbf{Z}^{n\times n}$.

In all cases $L \cong L_1 \oplus L_2$ with irreducible Λ-lattices L_1, L_2 satisfying $pL_1 \leq L_2 \leq L_1 \leq L_1^\# \leq p^{-1}L_1$, where the dual lattice is taken with respect to the positive definite generator of $\mathrm{Bil}_\Lambda(L_1)$. The individual cases are characterized by a chain of inclusions which can be read off from the rows of the matrices, like $L_1 = L_2 = L_1^\# = L_2^\#$ in the first case, $L_1 = L_2 < L_1^\# = L_2^\#$ in the second case, $pL_2^\# < L_2 < L_1 = L_1^\# < L_2^\#$ in the third case, or $pL_2^\# = L_2 < L_1 = L_1^\# < L_2^\#$ in the fourth case. Moreover, the second, the fourth, and the last three cases might have outer automorphisms.

To proceed to the promised structure theorem on $\mathrm{Out}_z^e(\mathrm{Bil}_\Lambda(L)$ for the depth 0 case, the following lemma is needed, which is implicit in [Neb98] and which certainly does not depend on the big Picard group machinery of [CuR87], Chapter 55.

LEMMA 3.8. *Let Γ be a hereditary order in a simple $\mathbf{Q}$-algebra $\mathcal{B}$, which has Schur index s and degree d (over its centre). Then $\mathrm{Out}_z(\Gamma)$ is Abelian. Moreover, if n is the number of primes in the centre $Z(\Gamma)$ dividing the discriminant of Γ with respect to the centre, then $\mathrm{Out}_z(\Gamma)$ can be embedded into an extension of the class group $Cl(Z(\Gamma))$ by $(C_{sd})^n$.*

Proof. Define $N(\Gamma) := \{b \in \mathcal{B}^* \mid b\Gamma b^{-1} = \Gamma\}$. Then $\mathrm{Out}_z(\Gamma) \cong N(\Gamma)/\langle\Gamma^*, Z(\mathcal{B})^*\rangle$. Let $\mathcal{W}$ be an irreducible $\mathcal{B}$-module. Then $N(\Gamma)$ acts on the Γ-sublattices in $\mathcal{W}$. For every prime $\mathfrak{p}$ in the centre of Γ, the $\Gamma_\mathfrak{p}$-sublattices in the completion $\mathcal{W}_\mathfrak{p}$ form a chain by inclusion, on which $N(\Gamma)$ acts by shifting the lattices up and down. Clearly the intersection of all the kernels of these shifts at the various primes is Γ^*. Hence $\mathrm{Out}_z(\Gamma)$ is Abelian.

More precisely, let $\mathrm{Sh}(\mathcal{W})$ be the group of all permutations of the Γ-sublattices of $\mathcal{W}$ which fixes all lattices in $\mathcal{W}_\mathfrak{p}$ for almost all primes $\mathfrak{p}$ in the centre of Γ and induces shifts at the remaining finitely many completions. Then $\mathrm{Sh}(\mathcal{W})$ is the direct sum of the $\mathrm{Sh}(\mathcal{W}_\mathfrak{p})$, each of which is infinite cyclic. Moreover $\mathrm{Sh}(\mathcal{W})$ acts regularly on the set of all nonzero Γ-lattices in $\mathcal{W}$. The above argument shows that $N(\Gamma)/\Gamma^*$ embeds into $\mathrm{Sh}(\mathcal{W})$. But so does the group F of all fractional ideals of $Z(\mathcal{B})$, resulting in a subgroup $\overline{F}$ of $\mathrm{Sh}(\mathcal{W})$. The cokernel of this embedding is isomorphic to a subgroup of a direct product of n cyclic groups, the order of each one of which divides sd. It is well known that $Z(\mathcal{B})^*$ maps into F with cokernel the class group of $Z(\Gamma)$ and kernel the torsion subgroup of $Z(\mathcal{B})$, which lies in Γ^* anyhow. Now by the above description of $\mathrm{Out}(\Gamma)$, it can be viewed as a subgroup of

$\mathrm{Sh}(\mathcal{W})/X$, where X is the image of $Z(\mathcal{B})^*$ in $\mathrm{Sh}(\mathcal{W})$. But $\mathrm{Sh}(\mathcal{W})/X$ is also an extension of $F/X \cong Cl(Z(\Gamma))$ by $\mathrm{Sh}(\mathcal{W})/\bar{F}$. □

As a consequence one gets the following

THEOREM 3.9. *Let* $\mathrm{Bil}_\Lambda(L)$ *be of depth* 0, *then* $\mathrm{Out}_z^e(\mathrm{Bil}_\Lambda(L))$ *is Abelian.* □

It is worthwhile to extract more precise statements from Lemma 3.8. They will be used and extended in the forthcoming chapter in the study of $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ when the e-$*$-depth of $\mathrm{Bil}_\Lambda(L)$ is zero.

DEFINITION 3.10. Let Γ be a hereditary $\mathbf{Z}$-order in a simple $\mathbf{Q}$-algebra $\mathcal{B}$ and let $\mathfrak{p}$ be a prime ideal in the centre $Z(\Gamma)$ of Γ. The $\mathfrak{p}$*-local shift index* $s(\Gamma,\mathfrak{p})$ of Γ is defined as follows: For any irreducible $\Lambda_\mathfrak{p}$-lattice L define $m(L)$ by $\mathfrak{p}^{m(L)} := [L : L_{max}]$, where L_{max} is the unique maximal Γ-sublattice of L. The chain $\cdots L_i \geq L_{i+1} \cdots$ of irreducible lattices in a simple $\mathcal{B}_\mathfrak{p}$-module $\mathcal{W}$ yields a periodic sequence $\ldots, m(L_i), m(L_{i+1}), \ldots$ because of $m(L) = m(\mathfrak{p}L)$. The index of the group of all "central" shifts generated by multiplication with $\mathfrak{p}$ in the group of all shifts of the chain respecting $m(L)$ is called $s(\Gamma, \mathfrak{P})$.

Obviously, $s(\Gamma,\mathfrak{p})$ is equal to the $\mathfrak{p}$-local Schur index of $\mathcal{B}$ if $\Gamma_\mathfrak{p}$ is a maximal order. In particular it is almost always equal to 1. With the definition of the local shift index at hand, the refined statement of Lemma 3.8, which was actually proved, should read as stated with $(C_{sd})^n$ replaced by $\bigoplus_\mathfrak{p} C_{c(\Gamma,\mathfrak{p})}$.

4. EXTRINSIC NOTIONS: USING THE UNDERLYING LATTICE

Up to now, the lattices $\mathrm{Bil}_\Lambda(L)$ of covariant forms have only been investigated by themselves without much reference to the underlying lattice L. In this section L will be taken more seriously into account. Unless confusion can arise L will also denote the underlying $\mathbf{Z}$-lattice of L, which is usually considered as a Λ-lattice.

To start with, we discuss the determinant function and its behaviour under equivalence.

DEFINITION 4.1.

$$\det\colon \mathrm{Bil}_\Lambda(L) \to \mathbf{Z} : \phi \mapsto \det(\phi_B)$$

is called the *determinant function* on $\mathrm{Bil}_\Lambda(L)$, where B is some lattice basis for L over $\mathbf{Z}$ and ϕ_B is the Gram matrix of ϕ with respect to B.

Clearly, choosing some $\mathbf{Z}$-basis for $\mathrm{Bil}_\Lambda(L)$ turns the determinant into a homogeneous polynomial in $\mathbf{Z}[X_1, \ldots, X_d]$ of degree $n = \dim_{\mathbf{Z}}(L)$ in $d = \dim_{\mathbf{Z}}(\mathrm{Bil}_\Lambda(L))$ variables. A connection of the factorization properties in $\mathbf{Q}[X_1, \ldots, X_d]$ with the structure of $\mathcal{V}$ is indicated in the rather obvious Remark 4.2 below. Those in $\mathbf{Z}[X_1, \ldots, X_d]$ have not yet been investigated. There sometimes seem to be changes in the factorization behaviour when one restricts from $\mathrm{Bil}_\Lambda(L)$ to $\mathrm{Bil}_\Lambda^+(L)$; cf. Chapter 5.

REMARK 4.2. Let $1 = e_1 + \ldots + e_h$ be the decomposition of $1 \in \mathcal{A}$ into central primitive idempotents of $\mathcal{A}$, and fix some isomorphism $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$. There is a constant $a = a(\psi, L) \in \mathbf{Q}$ depending on ψ such that, for all $\phi \in \mathrm{Bil}_\Lambda(L)$, one has

$$\det(\phi) = a \cdot \prod_{i=1}^{h} (\det{}_{\mathrm{red}}(\psi_i \phi_i))^{m(i)},$$

where the ϕ_i and ψ_i denote the restrictions of ϕ, resp. ψ, to $\mathcal{V}e_i$, resp. to $\mathcal{V}^* e_i$, $\det_{\mathrm{red}}$ is the reduced determinant of $\mathrm{End}_{\mathcal{A}}(\mathcal{V}e_i)$, and finally $m(i)$ is the degree of the matrix algebras which are the simple components of $\mathbf{C} \otimes_{\mathbf{Q}} \mathcal{A}e_i$.

If $\omega\colon \mathrm{Bil}_\Lambda(L) \to \mathrm{Bil}_\Lambda(M)$ is an equivalence, only the constant a in the above formula changes to some other constant $b = b(\omega'(\psi), M)$, and the exponents $m(i)$ change to the degrees $m(i)'$ of the corresponding simple components of $\mathbf{C} \otimes_{\mathbf{Q}} \mathcal{B}e_i'$. One has

$$\det(\omega(\phi)) = b \cdot \prod_{i=1}^{h} (\det{}_{\mathrm{red}}(\psi_i \phi_i))^{m(i)'},$$

since $\det_{\mathrm{red}}(\psi_i \phi_i) = \det_{\mathrm{red}}(\omega'(\psi_i)\omega(\phi_i))$, cf. discussion of Definition 2.6.

As an instructive example, which comes up as a step in the proof of Remark 4.2, the reader may want to relate the above formula to the well known determinant formula for the Kronecker product of two matrices.

DEFINITION 4.3. Let $(\mathcal{B}, ^\circ)$, $\mathcal{W}$, M, and R be as in Definition 2.6.

(i) Call L and M *form-R-equivalent*, or simply *form-equivalent* in case $R = \mathbf{Z}$, if there is an R-module isomorphism $\tau\colon M_R \to L_R$ which induces an R-equivalence $\omega\colon \mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(M_R) : \phi \mapsto \omega(\phi) = \tau\phi$ with $\tau\phi(W_1, W_2) = \phi(W_1\tau, W_2\tau)$ for all $W_1, W_2 \in M_R$. In this case $(L, \mathrm{Bil}_\Lambda(L))$ and $(M, \mathrm{Bil}_\Gamma(M))$ are also called R-equivalent and the R-equivalence ω is said to be *induced.*

(ii) We denote by $N(L_R)$ the group of all $\tau \in \mathrm{Aut}_R(L_R)$ inducing autoequivalences of $\mathrm{Bil}_\Lambda(L_R)$.

(iii) The group of all induced autoequivalences of $\mathrm{Bil}_{\Lambda_R}(L_R)$ is denoted $\mathrm{Aut}\big(\mathrm{Bil}_{\Lambda_R}(L_R)\big)$; its elements are also called *automorphisms of* $\mathrm{Bil}_\Lambda(L_R)$.

The connection with the earlier concepts is easily seen: for the determinant functions, one has $\det(\phi) = \det(\omega(\phi))$ for all $\phi \in \mathrm{Bil}_\Lambda(L)$ if the equivalence $\omega\colon \mathrm{Bil}_\Lambda(L) \to \mathrm{Bil}_\Gamma(M)$ is induced, i.e. the constant and the exponents in the formula of Remark 4.2 do not change any more. In other words, the associated polynomials in $\mathbf{Z}[X_1, \ldots, X_d]$ are $\mathbf{Z}$-equivalent, or even equal if one chooses appropriately the bases of the lattices of forms. Clearly, $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L)) \leq \mathrm{Aut}(\mathrm{Bil}_\Lambda(L)) \leq \mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ with all indices finite.

To get a full picture of the situation, one more group has to be introduced, namely the kernel of the epimorphism of $N(L)$ onto $\mathrm{Aut}(\mathrm{Bil}_\Lambda(L))$, which is $U(L)$ defined as follows.

DEFINITION 4.4.

(i) $U(L_R)$ is the image of the group $U\big(\Lambda(L)_R\big) := \{u \in \Lambda(L)_R \mid uu^\circ = 1\}$ in $\mathrm{Aut}_R(L_R)$ defined by its natural linear action on L_R.

(ii) The exact sequence

$$1 \to U(L) \to N(L) \to \mathrm{Aut}(\mathrm{Bil}_\Lambda(L)) \to 1$$

is called the *basic exact sequence.*

Obviously $U(L)$ is finite. If the $\mathbf{Q}$-algebra spanned by $U(L)$ is all of the image $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathrm{End}_{\mathbf{Q}}(\mathcal{V})$, then $N(L)$ is the normalizer of (the strict Bravais group) $U(L)$ in $\mathrm{Aut}_{\mathbf{Z}}(L)$; cf. [BNZ73]. In general one only has that $N(\mathcal{V})$ is the normalizer of $U(\mathcal{V})$ in $\mathrm{Aut}_{\mathbf{Q}}(\mathcal{V})$. The structure of $N(\mathcal{V})$ is easily worked out: it is dominated by the pair of semisimple subalgebras $\overline{\mathcal{A}}$ and $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ of $\mathrm{End}_{\mathbf{Q}}(\mathcal{V})$, which are centralizers of each other. In fact, if one restricts to the pointwise stabilizer $N_z(\mathcal{V})$ of the common centre of these two algebras, then $N_z(\mathcal{V})$ is the central product of $\mathrm{End}_{\mathcal{A}}(\mathcal{V})^*$ and a group $\widetilde{U}(\mathcal{V})$ amalgamated over their common centre, where $\widetilde{U}(\mathcal{V})$ is the image of $\{u \in \mathcal{A}^* \mid uu^\circ \in Z(\mathcal{A})\}$ in $\overline{\mathcal{A}}$. Note that the index $N(\mathcal{V}) : N_z(\mathcal{V})$ is finite. As a point of general notation, $\mathrm{Inn}(\Gamma)$ will always denote the group of automorphisms of a ring Γ induced by conjugating with units in Γ, and $\mathrm{Out}(\Gamma) := \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$.

PROPOSITION 4.5.

(i) *$N(L)$ acts on $\mathrm{Bil}_\Lambda(L)$ with kernel $U(L)$.*

(ii) *$N(L)$ acts on $\mathrm{End}_\Lambda(L)$ via conjugation also with kernel $U(L)$. In particular, $\mathrm{Aut}(\mathrm{Bil}_\Lambda(L))$ embeds into $\mathrm{Aut}(\mathrm{End}_\Lambda(L))$.*

(iii) *$N(L)$ acts on $\Lambda(L)$ by conjugation with kernel $\mathrm{End}_\Lambda(L)^*$. The induced automorphisms respect the involution $^\circ$.*

(iv) *Denote the kernel of the conjugation action of $N(L)$ on $Z(\mathrm{End}_\Lambda(L)) = Z(\Lambda(L))$ (or on $Z(\mathcal{A}) = Z\big(\mathrm{End}_{\mathcal{A}}(\mathcal{V})\big)$) by $N_z(L)$. Then $N_z(L)$ is a normal subgroup of finite index in $N(L)$ containing $\langle \mathrm{End}_\Lambda(L)^*, U(L)\rangle$, which is also of finite index.*

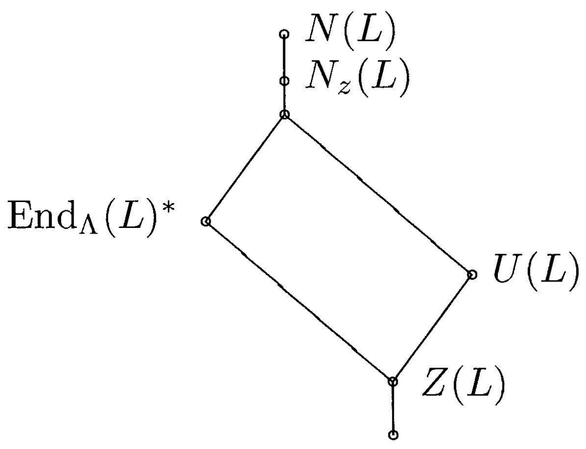
(v) *$\langle \mathrm{End}_\Lambda(L)^*, U(L)\rangle$ is a central product of $\mathrm{End}_\Lambda(L)$ and $U(L)$ amalgamated over $Z(L) := \mathrm{End}_\Lambda(L) \cap U(L)$.*

(vi) *The image of the conjugation action of $N_z(L)$ on $\Lambda(L)$ induces a finite index subgroup $\mathrm{Aut}_{z,L}(\Lambda(L), ^\circ)$ of $\mathrm{Aut}_z(\Lambda(L), ^\circ)$. The latter is also the image of the conjugation action of $\{u \in \widetilde{U}(\mathcal{A}, ^\circ) \mid u^{-1}\Lambda(L)u = \Lambda(L)\}$.*

(vii) *The image of the conjugation action of $N_z(L)$ on $\mathrm{End}_\Lambda(L)$ induces a subgroup $\mathrm{Aut}_{z,L}(\mathrm{End}_\Lambda(L))$ of $\mathrm{Aut}_z(\mathrm{End}_\Lambda(L))$. The latter is also the image of $\{\varphi \in \mathrm{End}_{\mathcal{A}}(\mathcal{V})^* \mid \varphi^{-1}\,\mathrm{End}_\Lambda(L)\varphi = \mathrm{End}_\Lambda(L)\}$*

(viii) *The group $N_z(L)/Z(L)$ is a subdirect product of $\mathrm{Aut}_{z,L}(\Lambda(L), ^\circ)$ and $\mathrm{Aut}_{z,L}(\mathrm{End}_\Lambda(L))$, amalgamated over the common finite factor group*

$$\mathrm{Aut}_{z,L}(\Lambda(L), ^\circ)/\mathrm{Inn}(\Lambda(L), ^\circ) \cong \mathrm{Aut}_{z,L}(\mathrm{End}_\Lambda(L))/\mathrm{Inn}(\mathrm{End}_\Lambda(L))$$
$$\cong N_z(L)/\langle \mathrm{End}_\Lambda(L)^*, U(L)\rangle\,.$$



Proof: Most of the statements can be verified in a straightforward way in the order in which they are listed, by using the preceding discussion of $N_z(\mathcal{V})$. The various finiteness statements follow from Proposition 3.5. □

Here is the main consequence for $\mathrm{Aut}(\mathrm{Bil}_\Lambda(L))$:

COROLLARY 4.6. $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L)) \trianglelefteq \mathrm{Aut}(\mathrm{Bil}_\Lambda(L))$, *and*

$$\mathrm{Out}(\mathrm{Bil}_\Lambda(L)) := \mathrm{Aut}(\mathrm{Bil}_\Lambda(L)) / \mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$$

embeds into $\mathrm{Out}(\mathrm{End}_\Lambda(L))$ *and into* $\mathrm{Out}(\Lambda(L), {}^\circ)$. *In particular,* $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ *is finite.*

It is worthwhile to extract the following slightly more technical consequence as well.

COROLLARY 4.7. *Denote by* $\mathrm{Aut}_z(\mathrm{Bil}_\Lambda(L))$ *the group of automorphisms of* $\mathrm{Bil}_\Lambda(L)$ *induced by* $N_z(L)$. *Then*

$$\mathrm{Inn}(\mathrm{Bil}_\Lambda(L)) \trianglelefteq \mathrm{Aut}_z(\mathrm{Bil}_\Lambda(L)) \trianglelefteq \mathrm{Aut}(\mathrm{Bil}_\Lambda(L)),$$

$\mathrm{Aut}(\mathrm{Bil}_\Lambda(L)) / \mathrm{Aut}_z(\mathrm{Bil}_\Lambda(L))$ *is isomorphic to a subgroup of the (obviously) finite group* $\mathrm{Aut}(Z(\Lambda(L))$, *and*

$$\mathrm{Out}_z(\mathrm{Bil}_\Lambda(L)) := \mathrm{Aut}_z(\mathrm{Bil}_\Lambda(L)) / \mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$$

embeds into the finite groups $\mathrm{Out}_z(\mathrm{End}_\Lambda(L))$ *and* $\mathrm{Out}_z(\Lambda(L), {}^\circ)$.

The next topics are the lattice versions of e-depth and e-$*$-depth, cf. Definition 3.6. Recall the notation introduced before Definition 3.6.

DEFINITION 4.8.

(i) Let $L^{(l)}$ be defined as $(\mathrm{End}_\Lambda(L))^{(l)}L$.

(ii) Define $L^{(0)} := L$ and $L^{(i)} := (L^{(i-1)})^{(l)}$, which yields an increasing sequence of full lattices in $\mathcal{V}$:

$$L = L^{(0)} \leq L^{(1)} \leq L^{(2)} \leq \cdots$$

(iii) The length of this sequence, i. e. the first i with $L^{(i)} = L^{(i+1)}$ is called the *depth* of L, resp. of $\mathrm{Bil}_\Lambda(L)$.

As a subtle point, note that $\mathrm{End}_\Lambda(L^{(1)})$ might contain $(\mathrm{End}_\Lambda(L))^{(1)}$ properly. In particular, L is of depth 0 if and only if $\mathrm{End}_\Lambda(L)$ is hereditary, which is also equivalent to $\mathrm{Bil}_\Lambda(L)$ having e-depth zero. For these situations the Picard group techniques mentioned above can easily be applied. But before going into the details of the depth-zero case, a general remark on the smoothing process must be made.

REMARK 4.9. $N(L^{(i)})$ acts on $L^{(i+1)}$, i. e. $N(L^{(i)})$ is conjugate to a subgroup of $N(L^{(i+1)})$ under $\mathrm{GL}(\mathcal{V})$.

Proof. Clearly, the conjugation action by elements in $N(L^{(i)})$ preserves $\mathrm{arad}(\mathrm{End}_\Lambda(L^{(i)}))$ and therefore also the idealizer $(\mathrm{End}_\Lambda(L^{(i)}))^{(l)}$. But $L^{(i+1)} = (\mathrm{End}_\Lambda(L^{(i)}))^{(l)} L^{(i)}$. □

Continuing the discussion of e-$*$-depth of the last section, the notion of $*$-depth will be defined. Ideally one is tempted to imitate Definition 4.8 along the following lines: define $L^{(rl)}$ as the lattice in $\mathcal{V}$ containing L with the property $\mathrm{End}_\Lambda(L \oplus L^*)^{(rl)}(L \oplus L^*) = L^{(rl)} \oplus M$ for some Λ-lattice M in $\mathcal{V}$ containing L^*; define $L^{[0]} := L$ and $L^{[i]} := (L^{[i-1]})^{(rl)}$, which yields an increasing sequence of full lattices in $\mathcal{V}$:

$$L = L^{[0]} \leq L^{[1]} \leq L^{[2]} \leq \cdots ;$$

and define the $*$-depth of L, resp. $\mathrm{Bil}_\Lambda(L)$, to be the length of this sequence, i. e. the first i with $L^{[i]} = L^{[i+1]}$.

To prove that everything is well defined, one needs a statement ensuring that this process really terminates. This boils down to: $\mathrm{End}_\Lambda\big(L^{(rl)} \oplus (L^{(rl)})^*\big)$ *contains* $\mathrm{End}_\Lambda(L \oplus L^*)$ *properly up to conjugation, unless* $\mathrm{End}_\Lambda(L \oplus L^*)$ *is hereditary.* I have not been able to prove this statement, though the argument below for the soundness of the less satisfactory definition, points somewhat in the right direction.

DEFINITION 4.10.

(i) Define sequences $L^{[0]} = L \leq L^{[1]} \leq L^{[2]} \ldots$ of lattices in $\mathcal{V}$ and $L^{*[0]} = L^* \leq L^{*[1]} \leq L^{*[2]} \ldots$ as follows:

$$L^{[i+1]} \oplus L^{*[i+1]} := \Gamma_i^{(rl)}(L^{[i]} \oplus L^{*[i]})$$

with $\Gamma_i := \mathrm{End}_\Lambda(L^{[i]} \oplus L^{*[i]}) \cap \mathrm{End}_\Lambda((L^{*[i]})^* \oplus (L^{[i]})^*)$.

(ii) Define the $*$-*depth* of L, resp. $\mathrm{Bil}_\Lambda(L)$, to be the length of these sequences, i. e. the first i with $L^{[i]} = L^{[i+1]}$ and $L^{*[i]} = L^{*[i+1]}$.

Here is a verification that the definition makes sense.

LEMMA 4.11.

(i) $\Gamma_i(L^{[i]} \oplus (L^*)^{[i]})$ *decomposes as indicated in Definition 4.10 (i).*

(ii) *For the order* Γ_i *one has* $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$, *so that the* $*$*-depth is well defined, namely as the first* i *with* Γ_i *hereditary.*

(iii) *Let* s *be the* $*$*-depth of* L, *then* $\mathrm{End}_\Lambda(L^{[s]} \oplus (L^{[s]})^*)$ *is hereditary, i. e. the* $*$*-depth of* $L^{[s]}$ *is zero.*

Proof. (i) Since the two idempotents mapping $L \oplus L^*$ onto L, resp. L^*, lie in any of Γ_i, the result follows.

(ii) By definition $\Gamma_i^{(lr)} \subseteq \mathrm{End}_\Lambda(L^{[i+1]} \oplus L^{*[i+1]})$. Moreover $\Gamma_i^{(lr)}$ is invariant under the involution; by Remark 2.4, it is therefore also contained in $\mathrm{End}_\Lambda((L^{*[i+1]})^* \oplus (L^{[i+1]})^*)$. Hence $\Gamma_i \subseteq \Gamma_i^{(lr)} \subseteq \Gamma_{i+1}$.

(iii) Γ_s is hereditary; hence $\mathrm{End}_{\Gamma_s}((L^{[s]} \oplus L^{*[s]}) \oplus ((L^{*[s]})^* \oplus (L^{[s]})^*)) \subset \mathcal{A}$ is hereditary. But $\Lambda(L^{[s]})$ contains this order and is therefore also hereditary, which makes $\mathrm{End}_\Lambda(L^{[s]} \oplus (L^{[s]})^*)$ hereditary. □

Various comments should be made. The notions of $*$-depth zero and e-$*$-depth zero are the same. This paper will mainly concentrate on the $*$-depth zero case, for which the two approaches yield the same answer. The first approach would in general be superior to the second one, because it defines a directed graph on the set of isomorphism classes of lattices in $\mathcal{V}$ with an arrow pointing from L to $L^{[1]}$ (in the first meaning).

This would have the nice property that one has no cycles except for the one with $*$-depth 0, and the depth of any lattice could be read off from the graph. In the second setting this is no longer possible. One has only an assignment to a $*$-depth zero lattice for any lattice without the intermediate steps. Example 2.2 (i) and Remark 2.8 show that one can produce situations where the $*$-depth is arbitrarily high with the depth being zero already.

It should be noted that this result implies a classical theorem by Watson, cf. [Wat62], which has been rediscovered by various people; and it puts the Watson process into the proper general framework. Strictly speaking, the assumption of positive definiteness is too strong, but it is retained here because it is the general hypothesis of the present paper. Various generalizations have been discarded, though they could have also been listed here.

COROLLARY (Watson). *Let* $L := \mathbf{Z}^{1\times n}$ *and* $\phi\colon L \times L \to \mathbf{Z}$ *be a (positive definite)* $\mathbf{Z}$*-bilinear form on* L*. Then there exists a full* $\mathbf{Z}$*-lattice* M *in* $\mathcal{V} := \mathbf{Q} \otimes_{\mathbf{Z}} L$ *which is* $\mathrm{Aut}(L, \phi)$*-invariant and satisfies* $M \subseteq M^{\#} \subseteq k^{-1}M$ *for some square-free divisor* k *of* $\det(L, \phi)$*, where the reciprocal lattice* $M^{\#}$ *is taken with respect to some rational multiple of* ϕ*.*

Proof. ϕ induces an involution on $\mathcal{A} := \mathbf{Q}^{n\times n}$ containing $\Lambda := \Lambda(L)$ as an invariant $\mathbf{Z}$-order. Denote the $*$-depth of L by s and set $M := L^{[s]}$. Clearly, $\mathrm{Aut}(L, \phi) = U(L)$, and M is $U(L)$-invariant. Since $\Gamma := \mathrm{End}_{\Lambda}(M \oplus M^*)$ is hereditary, the same applies to $\Lambda(M)$ $(= \mathrm{End}_{\Gamma}(M \oplus M^*))$. By the general properties of hereditary orders (as chain orders), the claim follows, since $M^{\#}$ is an absolutely irreducible Λ-lattice isomorphic to M^*. □

Having a canonical procedure for constructing $*$-depth zero lattices from ones of arbitrary $*$-depth such that the statements of Remark 4.9 carry over, it becomes an interesting question to look into the structure of $\mathrm{Out}_z(\mathrm{Bil}_{\Lambda}(L))$ in this case. Of course, it is no loss of generality if one restricts to the case of simple algebras $\mathcal{A}$. Here is a first statement, whose hypothesis is often satisfied.

THEOREM 4.12. *Let* L *be of* $*$*-depth zero and assume that the centre* $Z(\mathcal{A})$ *is a totally real number field. Then* $\mathrm{Out}_z(\mathrm{Bil}_{\Lambda}(L))$ *is of exponent dividing* 2.

Proof. Because of Proposition 4.5 (vi) and Corollary 4.7, one has to prove the following: for $u \in \widetilde{U}(\mathcal{A}) \cap N(\Lambda(L))$ the square u^2 induces an inner automorphism of $\Lambda(L)$. Let $uu^{\circ} = z$ for some element $z \in Z(\mathcal{A})$. Then u^2 and $z^{-1}u^2$ induce the same automorphism. But $z^{-1}u^2$ lies is $U(\mathcal{A})$, since $z^{\circ} = z$. Each prime of $Z(\mathcal{A})$ is mapped onto itself by the involution $^{\circ}$. Hence, at the completion of the whole situation at any prime $\mathfrak{p}$ of $Z(\mathcal{A})$, the element $z^{-1}u^2$ again lies in a unitary group and cannot induce a shift on the irreducible lattices in the sense of the proof of Lemma 3.8. It therefore lies in any completion of $\Lambda(L)$ and hence in $\Lambda(L)$. Since $\Lambda(L)$ is invariant under the involution, also the inverse of $z^{-1}u^2$ lies in $\Lambda(L)$ and the claim follows. □

Here is a $*$-depth zero example, where the hypothesis of Theorem 4.12 is violated and $\mathrm{Out}_z(\mathrm{Bil}_{\Lambda}(L))$ is of order 3.

EXAMPLE 4.13. Let $G := \langle a,b,c \mid a^7, b^3, a^b = b^2, c^3, [a,c], [b,c] \rangle$ be the group $(C_7 : C_3) \times C_3$ and let Λ be the residue class order of $\mathbf{Z}G$ modulo the ideal generated by $a-1$ and $c-1$. Then $\mathcal{A} \cong K^{3\times 3}$ with $K = \mathbf{Q}[\sqrt{-3}, \sqrt{-7}]$ (of class number 1, cf. [PoZ89]) and

$$\Lambda \cong \begin{pmatrix} R & R & R \\ I & R & R \\ I & I & R \end{pmatrix},$$

where $R = \mathbf{Z}[\frac{-1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-7}}{2}] = \mathbf{Z}_K$ is the maximal $\mathbf{Z}$-order in K and I is the product of the two prime ideals I_1 and I_2 above 7 in K, i.e. $7R = I^2$. The natural involution of $\mathbf{Q}G$ induces the involution $^\circ$ of $\mathcal{A}$ of interest. Finally, $L := \Lambda_\Lambda$ is chosen as the regular Λ-lattice, i.e. with respect to the above description of Λ, one has $L = (R,R,R) \oplus (I,R,R) \oplus (I,I,R)$. One easily checks that the group automorphism $a \mapsto a$, $b \mapsto bc$, $c \mapsto c$ maps Λ onto itself and things can be arranged so that (R,R,R) is mapped onto (I_1, R, I_2^{-1}), and (I_1,R,I_2^{-1}) onto $(I_1, I_1 I_2^{-1}, I_2^{-1})$. Since, clearly, $L \cong (R,R,R) \oplus (I_1,R,I_2^{-1}) \oplus (I_1, I_1 I_2^{-1}, I_2^{-1})$, this reveals an element of order 3 in $\mathrm{Aut}_z(\mathrm{Bil}_\Lambda(L))/\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$. In fact, $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ is of order 12.

The general situation for the $*$-depth zero case is as follows with the notation of Definition 3.10.

THEOREM 4.14. *Let L be of $*$-depth zero and assume (w.l.o.g.) that $\mathcal{A}$ is simple. Then* $\mathrm{Out}_z(\mathrm{Bil}_\Lambda(L))$ *is Abelian and embeds into an extension of the class group $Cl(Z(\Lambda(L)))$ of the centre $Z(\Lambda)$ by a group of the form*

$$\bigoplus_{\mathfrak{p}\in\mathcal{S}} C_2 \oplus \bigoplus_{\{\mathfrak{p},\mathfrak{p}^\circ\}\in\mathcal{N}} C_{s(\Lambda(L),\mathfrak{p})},$$

with $\mathcal{S}$ the set of prime ideals $\mathfrak{p}$ of $Z(\mathcal{A})$ with $\mathfrak{p} = \mathfrak{p}^\circ$ and $s(\Lambda(L),\mathfrak{p})$ even, and $\mathcal{N}$ the set of pairs $\{\mathfrak{p},\mathfrak{p}^\circ\}$ of prime ideals with $\mathfrak{p} \neq \mathfrak{p}^\circ$.

Proof. That $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ is Abelian was already shown in Theorem 3.9. As in the proof of Theorem 4.12, let $u \in \widetilde{U}(\mathcal{A}) \cap N(\Lambda(L))$. At each prime $\mathfrak{p}$ of $Z(\mathcal{A})$, u induces a shift of period $a(\mathfrak{p}) \mid s(\Lambda(L),\mathfrak{p})$, as explained in Lemma 3.8 and Definition 3.10. Let $uu^\circ = z$ for some element $z \in Z(\mathcal{A})$. At the real primes $\mathfrak{p} = \mathfrak{p}^\circ$, both u and u° shift by the same index, and hence the induced shift generates at most a subgroup of order 2 of $C_{s(\Gamma(L),\mathfrak{p})}$. If $\mathfrak{p} \neq \mathfrak{p}^\circ$, the induced shifts at $\mathfrak{p}$ and $\mathfrak{p}^\circ$ are opposite to each other and of the same order modulo local central shifts. Since the situation is global, the class group of the centre has to be taken into account, as in the proof of Lemma 3.8. □

5. Inversion and Modularity

Since $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \subseteq \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*)$, the inverse ϕ^{-1} of a nondegenerate $\phi \in \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ is well defined and lies in $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$. By Cramer's rule inversion is a rational map from $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ to $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$, more precisely there is a homogeneous polynomial map $P\colon \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \to \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$ such that $\phi^P\phi = \det(\phi)\cdot id_{\mathcal{V}}$. Viewing this as an identity of matrices with polynomial entries, one might cancel out the greatest common divisor of all occurring entries and get new polynomial maps $p\colon \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \to \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$ and $d\colon \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \to \mathbf{Q}$ with $\phi^p\phi = d(\phi)\cdot id_{\mathcal{V}}$. The properties of the map p have not been studied in this generality. The aim here is to investigate the simplest case, where p is homogeneous of degree 1, i. e. a $\mathbf{Q}$-linear map ι, as it is called in the sequel. Of course, the same analysis can be done with $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$. The question whether such a ι is an equivalence, will be treated later in this section.

Definition 5.1. Let R be one of $\mathbf{Z}$ or $\mathbf{Q}$. Then $\mathrm{Bil}_{\Lambda_R}(L_R)$ is called *special* if there is an R-linear map $\iota\colon \mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(L_R^*)$ and a quadratic form $q\colon \mathrm{Bil}_{\Lambda_R}(L_R) \to R$ such that for any nondegenerate $\phi \in \mathrm{Bil}_{\Lambda_R}(L_R)$ one has $\phi^\iota\phi = q(\phi)\, id_{L_R}$. Analogous definitions hold for $\mathrm{Bil}^+_{\Lambda_R}(L_R)$

Example 5.2.

(i) One-dimensional lattices of covariant forms are special for trivial reasons.

(ii) If $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ is two-dimensional, then it is special. This is because $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ can be viewed as a free $Z(\mathcal{A})$-module and for two-dimensional algebras $\mathcal{B}$ one has a canonical automorphism κ of $\mathcal{B}$ such that $b^\kappa = n(b)b^{-1}$ for all $b \in \mathcal{B}^*$, where $n\colon \mathcal{B} \to F$ is the norm map with respect to the regular representation. (Note that $Z(\mathcal{A}) = \mathrm{End}_{\mathcal{A}}(\mathcal{V})$ in the present situation.)

(iii) If $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ is two-dimensional then it is special. This is because $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ can be viewed as a free $Z(\mathcal{A})^+$-module, where

$$Z(\mathcal{A})^+ := \{\varphi \in Z(\mathcal{A}) \mid \varphi^\circ = \varphi\}.$$

Here are some more interesting examples.

Proposition 5.3. *Let* $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2\times 2}$ *with* $K \in \{\mathbf{R},\mathbf{C},\mathbf{H}\}$. *Then* $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ *is special. In the first two cases also* $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ *is special.*

Proof. Define $\mathcal{E} := \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong (e\mathcal{A}e)^{k\times k}$, where $e = e^{\circ}$ is a primitive $^{\circ}$-invariant idempotent of $\mathcal{A}$ and k is defined by $\mathcal{V} \cong (e\mathcal{A})^k$. In particular, the positive involution $^{\circ}$ on $\mathcal{A}$ induces a positive involution $^{\bullet}$ on $\mathcal{E}$, $(a_{ij})^{\bullet} := (a_{ij}^{\circ})^{tr}$, such that $\mathrm{Bil}_{\mathcal{A}}^{+}(\mathcal{V})$ can be identified with the subspace $\mathcal{E}^{+}$ of the symmetric elements in the algebra $(\mathcal{E}, {}^{\bullet})$ with involution. It suffices to prove that there exists a $\mathbf{Q}$-vector space automorphism of $\mathcal{E}^{+}$, also denoted by ι, and a $\mathbf{Q}$-valued quadratic form on $\mathcal{E}^{+}$, also denoted by q, such that $\phi^{\iota}\phi = q(\phi)\, 1_{\mathcal{E}}$.

(i) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{R}^{2\times 2}$. Then $\mathcal{E}$ is a quaternion algebra over $\mathbf{Q}$. Denote its canonical involution by ω' and its reduced norm by n. Clearly, n is a quadratic form and $\omega'(\phi)\ \phi = n(\phi)\, 1$ holds for all elements $\phi \in \mathcal{E}$. With $\iota := \omega'_{|\mathcal{E}^{+}}$ and $q := n_{|\mathcal{E}^{+}}$ one gets the desired formula.

(ii) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{C}^{2\times 2}$. Then $\mathcal{E}$ is a quaternion algebra over the imaginary quadratic number field $Z := Z(\mathcal{A})$. Denote its canonical involution by ω' and its reduced norm by n. The involution $^{\bullet}$ induces the nontrivial Galois automorphism of $(Z/\mathbf{Q})$, and therefore one checks quite easily, using [Scha85] Theorem 11.2 (ii) of Chapter 8, that the norm n maps $\mathcal{E}^{+}$ into $\mathbf{Q}$. Now one argues as in (i).

(iii) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{H}^{2\times 2}$. Then $\mathcal{E} \cong D^{2\times 2}$, where D is a positive definite quaternion algebra over $\mathbf{Q}$ (with canonical involution ω'). Indeed, $\mathcal{E}$ carries an involution of the first kind and hence cannot be of index 4. Since $^{\bullet}$ is a positive involution one sees from the proof of Theorem 13.3 of Chapter 8 in [Scha85] that $x^{\bullet} = f^{-1}\overline{x}^{tr} f$ for all $x \in \mathcal{E}$, where $f = \overline{f}^{tr} \in \mathcal{E}^{*}$ and $\overline{(x_{ij})} = (\overline{x_{ij}})$ for all $(x_{ij}) \in D^{2\times 2} \equiv \mathcal{E}$. If $(x_{ij}) \in \mathcal{E}$ is symmetric with respect to $^{-tr}$ one checks

$$(x_{ij}) = \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} \quad \text{with } \overline{x_{ii}} = x_{ii} \text{ for } i = 1, 2$$

$$\text{and} \quad \begin{pmatrix} x_{22} & -x_{12} \\ -\overline{x_{12}} & x_{11} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} = (x_{22}\, x_{11} - x_{12}\, \overline{x_{12}})\, 1_{\mathcal{E}}\, .$$

This is the desired formula for $f = 1_{\mathcal{E}}$. In the general case, note that $x \in \mathcal{E}^{+}$ if and only if fx is symmetric with respect to $^{-tr}$ and apply the above formula to fx.

(iv) The remaining two cases for $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ are treated similarly, like (i) and (ii) with $\mathcal{E}^{+}$ replaced by $\mathcal{E}$. □

The question immediately arises, whether the map ι of Definition 5.1 is or can be extended to an equivalence of $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ onto $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V}^{*})$. This is clearly the case for two-dimensional $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$. It may fail for two-dimensional $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ with four-dimensional commutative $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ for the simple reason that the nontrivial automorphism of the real quadratic subfield does not necessarily extend to the whole of $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$. For $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{R}^{2\times 2}$ one gets a nice canonical answer, cf. Proposition 5.4 below. For $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{C}^{2\times 2}$ the answer is still positive, but the proof is computational and we omit it. Finally, for $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{H}^{2\times 2}$ the map ι no longer extends to an equivalence.

PROPOSITION 5.4. *Let $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{R}^{2\times 2}$. Then any nonzero $\psi\in\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V}^*)$ defines an equivalence $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})\to\mathrm{Bil}_{\mathcal{A}}(\mathcal{V}^*):\phi\mapsto\psi\phi\psi^{tr}$ which restricts to a map $\iota\colon\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})\to\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$ with the properties described in Proposition 5.3.*

Proof. If $\mathcal{V}$ is a simple $\mathcal{A}$-module, obviously any nonzero element of $\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V}^*)$ is invertible if viewed as an $\mathcal{A}$-homomorphism from $\mathcal{V}^*$ to $\mathcal{V}$. Otherwise, $\mathcal{V}\cong\mathcal{V}_0\oplus\mathcal{V}_0$ for some simple $\mathcal{A}$-module $\mathcal{V}_0$. Any $\mathcal{A}$-isomorphism $\mathcal{V}_0\to\mathcal{V}_0^*$ gives rise to an invertible element of $\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V})$, which therefore consists of 0 and invertible elements, since it is one-dimensional. One easily checks that any nonzero $\psi\in\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V}^*)$ leads to an equivalence, whose associated isomorphism $\mathrm{End}_{\mathcal{A}}(\mathcal{V}\oplus\mathcal{V}^*)\to\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*\oplus\mathcal{V})$ is induced by conjugation with $\mathrm{diag}(-\psi^{-1},\psi)$. Finally, for any $\phi\in\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ one has $\phi(\psi\phi\psi^{tr})=q(\phi)id_{\mathcal{V}}$ with $q(\phi):=n(\psi\phi)$, where n is the reduced norm map of the quaternion algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$. This is so, since $\phi(\psi\phi\psi^{tr})=-(\phi\psi)^2$ and $\phi\psi$ lies in $\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$ and is of trace zero by $tr(\phi\psi)=tr((\phi\psi)^{tr})=tr(-\psi\phi)=-tr(\phi\psi)$. $\square$

The next result normalizes ι and interprets it in the integral environment of $\mathrm{Bil}^+_{\Lambda}(L)$.

THEOREM 5.5. *Let $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong K^{2\times 2}$ with $K\in\{\mathbf{R},\mathbf{C},\mathbf{H}\}$.*

(i) *There is a unique $\mathrm{Aut}(\mathrm{Bil}_{\Lambda}(L))$-invariant quadratic form $q\colon\mathrm{Bil}^+_{\Lambda}(L)\to\mathbf{Z}$ such that the $\gcd(q(\phi))$ for $\phi\in\mathrm{Bil}^+_{\Lambda}(L)$ is 1, and $q(\phi)>0$ for $\phi\in\mathrm{Bil}^+_{\Lambda}(L)$ positive definite.*

(ii) *There is a unique constant $c\in\mathbf{Z}$ satisfying $\det(\phi)=cq(\phi)^m$ with $m=2^{-1}\dim_{\mathbf{Q}}\mathcal{V}$ for all $\phi\in\mathrm{Bil}^+_{\Lambda}(L)$. (Clearly $c\geq 1$.)*

(iii) *There is a unique $\mathrm{Aut}(\mathrm{Bil}_{\Lambda}(L))$-monomorphism $\iota\colon\mathrm{Bil}^+_{\Lambda}(L)\to\mathrm{Bil}^+_{\Lambda}(L^*)$ mapping positive definite forms on positive definite ones such that the image of ι is not contained in $p\,\mathrm{Bil}^+_{\Lambda}(L^*)$ for any integer $p\geq 2$.*

(iv) *There is a unique constant* $c_0 \in \mathbf{Z}$ *with* $\phi^\iota\phi = c_0 q(\phi) id_L$ *for all* $\phi \in \mathrm{Bil}_\Lambda^+(L)$. *Moreover* c *divides* c_0^n, *where* $n = \dim_{\mathbf{Q}} \mathcal{V}$. *(In fact* $\det(\phi^\iota) = c_0^n c^{-1} q(\phi)^m$ *for all* $\phi \in \mathrm{Bil}_\Lambda^+(L)$.*)*

(v) $\mathrm{Aut}\big(\mathrm{Bil}_\Lambda^+(L)\big) \leq \mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$ *is a subgroup of finite index.*

Proof. Let $\mathrm{Bil}_\Lambda^+(L) = \langle \phi_1, \phi_2, \ldots, \phi_d \rangle_{\mathbf{Z}}$ (with $d = 3, 4$, resp. 6 for $K = \mathbf{R}$, $\mathbf{C}$, resp. $\mathbf{H}$). Choose the isomorphism ι of Proposition 5.3 by multiplying with a suitable positive rational number such that $\mathrm{Bil}_\Lambda^+(L)$ is mapped into $\mathrm{Bil}_\Lambda^+(L^*)$ but not into a proper multiple of $\mathrm{Bil}_\Lambda^+(L^*)$. After rescaling q of Proposition 5.3 appropriately, one gets a quadratic form $\widetilde{q} \in \mathbf{Z}[x_1, \ldots, x_d]$ with

$$\Big(\sum_{i=1}^{d} x_i\phi_i^\iota\Big)\Big(\sum_{i=1}^{d} x_i\phi_i\Big) = \widetilde{q}(x_1, \ldots, x_d)\, id_L\,.$$

Since $\mathbf{Z}[x_1, \ldots, x_d]$ is a unique factorization domain, one obtains a constant c_0 and a quadratic form q as required in (i) and (iv). Also by taking determinants, the unique factorization property yields $\det(\phi) = cq(\phi)$ with a unique integer c dividing c_0^n. Since $\det(g\phi) = \det(g)^2 \det(\phi) = \det(\phi)$ for $g \in N(L)$, one sees that q is $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$-invariant, at least up to sign. And since the action respects positive definiteness, one gets invariance. One clearly has $(g\phi)^\iota = g^{-tr}\phi^\iota$ for all $g \in N(L)$ and all $\phi \in \mathrm{Bil}_\Lambda^+(L)$ of nonzero determinant. But since all other elements of $\mathrm{Bil}_\Lambda^+(L)$ are rational linear combinations of these, one obtains the equation for all $\phi \in \mathrm{Bil}_\Lambda^+(L)$.

To prove (v) we first note that, by a standard Lie group argument, the group S of norm 1 units of $\mathrm{End}_{\mathcal{A}}(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V})$ is mapped onto the 1-component of $\mathrm{O}(\mathrm{Bil}_{\mathbf{R}\otimes\Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$. Also it is well known that the subgroup Γ of norm 1 elements of $\mathrm{End}_\Lambda(L)^*$ (which is clearly of finite index in $N(L)$) has finite covolume in S. This implies that $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ is of finite covolume in $\mathrm{O}(\mathrm{Bil}_{\mathbf{R}\otimes\Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$ and therefore of finite index in $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$.

It follows from (v) and the fact that the signature of q is $(1, d-1)$ that $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ acts absolutely irreducibly on $\mathrm{Bil}_\Lambda^+(L)$. This again implies that the invariant quadratic form q is unique up to rational multiples, i. e. unique with the properties specified in (i). It also implies the uniqueness of ι in (iii). The uniqueness of the constants c_0 and c now follows from the considerations at the beginning of the proof. $\square$

The corresponding results for the other examples given in Example 5.2 are left as exercises to the reader, who should note however that the action of $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$ on $\mathrm{Bil}_\Lambda^+(L)$ need not be absolutely irreducible any more.

The next topic it to set the concepts of this chapter into relation with modular lattices as introduced by Quebbemann in [Que95]; cf. also [SSch98] and [Ple98] for surveys.

DEFINITION 5.6.

(i) $\phi \in \mathrm{Bil}^+_\Lambda(L)$ is said to be *k-modular*, for $k \in \mathbf{Z}$, if $(L^\sharp, k\phi)$ is isometric to (L, ϕ), where $L^\sharp = \{l \in \mathcal{V} \mid \phi(l, L) \subseteq \mathbf{Z}\}$. (Note the Gram matrix of ϕ on $L^\sharp$ is inverse to the Gram matrix on L if one chooses the bases dual to each other.)

(ii) $\mathrm{Bil}^+_\Lambda(L)$ is called *modular* if $\mathrm{Bil}^+_\Lambda(L)$ is special by the maps $\iota\colon \mathrm{Bil}^+_\Lambda(L) \to \mathrm{Bil}_\Lambda(L^*)$ and $q\colon \mathrm{Bil}^+_\Lambda(L) \to \mathbf{Z}$, cf. Definition 5.1, such that ι is (the restriction to $\mathrm{Bil}^+_\Lambda(L)$ of) an induced equivalence; cf. Definition 4.3.

Clearly, if $\mathrm{Bil}^+_\Lambda(L)$ is modular, each nondegenerate $\phi \in \mathrm{Bil}^+_\Lambda(L)$ is $c_0 q(\phi)$-modular with c_0 as in Theorem 5.5, and the isometries are all given by the same map. Some examples of two-dimensional modular lattices of covariant forms have already been investigated in the literature, cf. e. g. [Neb98b] where even the Hermite function was discussed for some examples or [Neb96a], where the extremal 3-modular lattice in dimension 24 was discovered. Here the main issue concerns the cases with $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2\times 2}$ or $\mathbf{C}^{2\times 2}$, since $\mathbf{H}^{2\times 2}$ cannot occur. Example 6.6 (i) provides an example where $\mathrm{Bil}^+_\Lambda(L)$ is special without being modular. It should be emphasized that induced equivalence between $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Lambda(L^*)$ is not an uncommon phenomenon. For instance it occurs whenever L and L^* are Λ-isomorphic. That the induced equivalence is ι, is rather rare.

PROPOSITION 5.7. *Let* $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2\times 2}$ *and assume* $\mathrm{Bil}^-_\Lambda(L) = \mathbf{Z}\psi_1$ *and* $\mathrm{Bil}^-_\Lambda(L^*) = \mathbf{Z}\psi_2$ *with* $\psi_1\psi_2 = e \cdot id_L$ *for some natural number* e.

(i) *If* $e = 1$ *then* $\mathrm{Bil}_\Lambda(L)$ *is modular, with* ι *induced by* ψ_2.

(ii) *If* ψ_1 *and* ψ_2 *do not have the same elementary divisors, then* $\mathrm{Bil}^+_\Lambda(L)$ *is not modular.*

(iii) *If* $e^{\dim(L)} \neq \det(\psi_2)^2$ *then* $\mathrm{Bil}^+_\Lambda(L)$ *is not modular.*

Proof. (i) This follows along the lines of Proposition 5.4. That $\mathrm{Bil}_\Lambda(L)$ is mapped onto $\mathrm{Bil}_\Lambda(L^*)$ follows from the fact that $\det(\psi_2) = \pm 1$.

(ii) This is because induced equivalence respects elementary divisors.

(iii) This can be derived from (ii) by taking determinants. It can also be obtained from the observation that ψ_2 induces $e \cdot \iota$. □

EXAMPLE 5.8.

(i) Of the four irreducible Bravais groups of degree 8 whose commuting algebra is a nonsplit rational quaternion algebra (ramified at 2 and 3), cf. [Sou94], the e of Proposition 5.7 is 1, 2, 3 and 6. In all cases $\mathrm{Bil}_{\Lambda}^{+}(L)$ is modular and c_0 is equal to 1. In [Neb99] the Hermite function on the fundamental domains for these cases is plotted.

(ii) In Example 2.2 (ii), choose f_0 to be m-modular for some natural number m. Then $\mathrm{Bil}_{\Lambda}^{+}(L \oplus L)$ (in the notation of Example 2.2 (ii)) is modular, where the e of Proposition 5.7 is equal to m, as is c_0.

To test whether $\mathrm{Bil}_{\Lambda}^{+}(L)$ is modular, one can simply compute the images of a $\mathbf{Z}$-basis of $\mathrm{Bil}_{\Lambda}^{+}(L)$ under ι as described in Theorem 5.5 and find a simultaneous isometry of L to L^* (with respect to all of the forms, resp. their images). For this there is a powerful algorithm with implementation available, cf. [PlS97]. Instead of a whole basis, it is sometimes enough to look at one sufficiently general form; details on this will be given in a subsequent paper, as well as some examples with $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2\times 2}$. One such example, involving the Leech lattice with $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ a non-split quaternion algebra over $\mathbf{Q}[\sqrt{-7}]$, is sketched in the last chapter of [Ple96].

6. SOME THREE-DIMENSIONAL LATTICES OF COVARIANT FORMS

This chapter is devoted to some examples in the case where $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$ and where the depth of $\mathrm{Bil}_{\Lambda}(L)$ is 0. The typical questions we try to answer are: how to relate the various invariants? are outer automorphisms possible? are modular lattices possible? how does the automorphism group of $\mathrm{Bil}_{\Lambda}^{+}(L)$ compare to the orthogonal group of $(\mathrm{Bil}_{\Lambda}^{+}(L), q)$? The simplest case is $\mathrm{End}_{\Lambda}(L) \cong \mathbf{Z}^{2\times 2}$, where all these questions can be answered.

THEOREM 6.1. *Let* $\mathrm{End}_{\Lambda}(L) \cong \mathbf{Z}^{2\times 2}$. *Then* $L = L_0 \oplus L_0$ *for some irreducible* Λ*-lattice* L_0. *Let* ϕ_0 *be the positive definite generator of* $\mathrm{Bil}_{\Lambda}^{+}(L_0)$. *Then* c, c_0, *and* q, *introduced in Theorem 5.5, are as follows.*

(i) *With respect to a suitable basis of* $\mathrm{Bil}_{\Lambda}^{+}(L)$, *the quadratic form* q *of Theorem 5.5 becomes* $xy - z^2$.

(ii) $c = \det(\phi_0)^2$.

(iii) c_0 *is the exponent of* $L_0^{\sharp}/L_0$, *i. e. the biggest elementary divisor of a Gram matrix of* ϕ_0.

(iv) $\mathrm{Inn}(\mathrm{Bil}_\Lambda^+(L)) = \mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$.

(v) $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ *is of index* 2 *in* $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$. *More precisely, it is equal to the kernel of* $-\theta$ *intersected with* $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L))$, *where* θ *is the spinor norm of* $\mathrm{O}(\mathrm{Bil}_\mathcal{A}^+(\mathcal{V}), q)$ *([Scha85], p. 336).*

(vi) *The nondegenerate* $\phi \in \mathrm{Bil}_\Lambda^+(L)$ *are modular if and only if* ϕ_0 *is* c_0*-modular. In this case such a* ϕ *is* $c_0 q(\phi)$*-modular.*

(vii) *The e-$*$-depth of* $\mathrm{Bil}_\Lambda(L)$ *is given by* $[\frac{r}{2}]$, *where* r *is maximal with* $p^r \mid c_0$ *for some prime number* p.

Proof. Choose a basis for L_0. This yields a Gram matrix A of ϕ_0. With respect to a suitable basis of L, one gets $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ as Gram matrices for the obvious basis of $\mathrm{Bil}_\Lambda^+(L)$. Since $\det\left(\begin{pmatrix} x & z \\ z & y \end{pmatrix} \otimes A\right) = \det(A)^2(xy - z^2)^m$ and $\left(\begin{pmatrix} x & z \\ z & y \end{pmatrix} \otimes A\right)^{-1} = (xy - z^2)^{-1}\begin{pmatrix} y & -z \\ -z & x \end{pmatrix} \otimes A^{-1}$, the claims (i) to (iv) follow. (v) is straightforward with [Mac81]. (vi) and (vii) are obvious. □

The general case of depth 0 is more involved:

PROPOSITION 6.2. *Assume* $\mathcal{E} \cong \mathbf{Q}^{2\times 2}$ *and* L, *resp.* $\mathrm{Bil}_\Lambda^+(L)$, *is of depth* 0. *Let* $d := p_1 \cdots p_k$ *be the product of the different primes at which* $\mathrm{End}_\Lambda(L)$ *is not maximal.*

(i) *There are unique natural numbers* s, t *such that the quadratic form* q *on* $\mathrm{Bil}_\Lambda^+(L)$ *described in Theorem 5.5 becomes* $sxy - tz^2$ *with respect to any basis* (ϕ, ψ, χ) *of* $\mathrm{Bil}_\Lambda^+(L)$ *such that* $\phi, \psi \in \mathrm{Bil}_{\Lambda,\geq 0}^+(L)$ *with* $L = Rad_\psi(L) \oplus Rad_\phi(L)$ *and* χ *is zero on both direct summands. The product* st *divides* d.

(ii) *The constant* c *of Theorem 5.5 is given by*

$$c = \det(\overline{\phi}) \det(\overline{\psi}) s^{-m},$$

where $2m = \dim_{\mathbf{Q}}(\mathcal{V})$, $\overline{\phi}$ *is the scalar product on* $Rad_\psi(L)$ *induced by* ϕ, *and* $\overline{\psi}$ *the scalar product on* $Rad_\phi(L)$ *induced by* ψ.

Note that, providing $k > 0$, there are 2^{k-1} such bases up to interchanging ϕ and ψ and up to $\mathrm{End}_\Lambda(L)$ operation.

Proof. Let $L = L_1 \oplus L_2$ with absolutely irreducible Λ-lattices L_1, L_2. One may assume $dL_1 \leq L_2 \leq L_1$. Note this implies that L_1^* can be considered to sit inside L_2^* with $L_1^* \leq L_2^* \leq d^{-1}L_1^*$. As a result, $\mathrm{Hom}_\Lambda(L_1, L_2^*) = d_1 \mathrm{Hom}_\Lambda(L_1, L_1^*)$ for some divisor d_1 of d, and

$\mathrm{Hom}_\Lambda(L_2, L_2^*) = d_2 \mathrm{Hom}_\Lambda(L_1, L_2^*)$ for some divisor d_2 of d. Introducing a basis for L_1, as with L_0 in Theorem 6.1, identifies $L_1 =: L_0$ with $\mathbf{Z}^{1\times m}$; and choosing a basis for L_2 identifies L_2 with $\mathbf{Z}^{1\times m}T$, where $T \in \mathbf{Z}^{m\times m}$ represents the change of bases. Denote the $m \times m$-unit matrix by $I = I_m$. The computation for Theorem 6.1 can be transformed as follows:

$$\begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} x'A & z'A \\ z'A & y'A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix}^{tr} = \begin{pmatrix} xA & zd_1^{-1}AT^{tr} \\ zd_1^{-1}TA & yd_1^{-1}d_2^{-1}TAT^{tr} \end{pmatrix},$$

with $x = x', z = d_1 z', y = d_1 d_2 y'$. The parameter choice $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ for (x,y,z) yields a typical basis for $\mathrm{Bil}_\Lambda^+(L)$ as described above. Taking determinants yields

$$\det(T)^2 \det(A)^2 \Big(\frac{xy}{d_1 d_2} - (\frac{z}{d_1})^2\Big)^m,$$

and hence (i) and (ii) with $s = d_1 g^{-1}, t = d_2 g^{-1}$ relatively prime, where $g = \gcd(d_1, d_2)$, if one uses $\det(\overline{\phi}) = \det(A)$. That s, t do not depend on the particular decomposition of L follows from analyzing the determinant of q. $\square$

Working through the various cases for determining c_0 in Theorem 5.5 is left as an exercise. Before analyzing $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ one needs to look at the automorphism groups of the quadratic forms involved. Note that the automorphism groups of $kxy - z^2$ for $k \in \mathbf{N}$ square free are analyzed in quite some detail in [Mac81]. In the present context two extra details are needed.

LEMMA 6.3. *Let $s, t \in \mathbf{N}$ be square free and relatively prime, and let $k := st$.*

(i) *The diagonal matrix* $\mathrm{diag}(t,t,1)$ *transforms* $O(\mathbf{Z}^{1\times 3}, sxy - tz^2)$ *onto* $O(\mathbf{Z}^{1\times 3}, kxy - z^2)$.

(ii) *There is an exact sequence of groups:*

$$\langle -I_2 \rangle \hookrightarrow \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z} \end{pmatrix}^* \to O(\mathbf{Z}^{1\times 3}, kxy - z^2) \to D_k \to 1,$$

where $D_k \leq \mathbf{Q}^*/(\mathbf{Q}^*)^2$ is generated by the cosets of the divisors d of k (including -1).

Proof. (i) Denote the quadratic forms $sxy - tz^2$ and $kxy - z^2$ by q and q' respectively. On $L = \mathbf{Z}^{1\times 3}$ they define integral bilinear forms b and b', e.g. $b(l_1, l_2) = q(l_1 + l_2) - q(l_1) - q(l_2)$ for $l_1, l_2 \in L$. Clearly, $O(L, q)$ also acts on the reciprocal lattice $L^\sharp$ of L with respect to b, and $O(L, q')$ also acts on the reciprocal lattice $L'^{\sharp}$ of L with respect to b'. Hence $\mathrm{diag}(t,t,1)$, which maps L onto $tL^{\sharp}\cap L$ and q onto tq', conjugates $\mathrm{O}(L,q)$ into $\mathrm{O}(L,q')$. For the reverse inclusion one argues similarly for t odd with $tL'^{\sharp}\cap L$ and one has to work with $\frac{t}{2}L'^{\sharp}\cap L$, taking the even sublattice, for t even.

(ii) Define $L_d := \{\left(\begin{smallmatrix} a & c \\ c & db\end{smallmatrix}\right) \mid a,b,c\in\mathbf{Z}\}$ and consider the determinant det as a quadratic form on L_d for any natural number d. Then $(L_k,\det)$ is isometric to $(\mathbf{Z}^{1\times 3}, kxy - z^2)$. One easily checks that $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$ acts on L_k by $X\mapsto gXg^{tr}$ for all $X\in L_k$ and $g\in\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$. Clearly this action respects the determinant, whence the exactness of the left half of the sequence is established. Note, for $k=1$, the full claim was already proved in Theorem 6.1. Clearly $L_k\le L_1$ and the stabilizer S_k of L_k in $\mathrm{O}(L_1,\det)$ is generated by $-id_{L_1}$ and the image of $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$. As in Theorem 6.1 denote the spinor norm of $\mathrm{O}(\mathbf{Q}^{1\times 3}, xy-z^2)$ by θ. Then $-\theta$ restricted to $\mathrm{O}(L_k,\det)$ will be the homomorphism on the right of the exact sequence. Clearly the image of $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$ is in the kernel of $-\theta$. To complete the proof, it is enough to show, by induction on the number $d(k)$ of prime divisors of k, that $\mathrm{O}(L_k,\det)$ contains S_k of index $2^{d(k)}$ and is generated by an S_k and elements (Atkin-Lehner involutions) mapped by $-\theta$ onto $p(\mathbf{Q}^*)^2$ for the primes p dividing k.

The statement follows for $d(k)=1$, i.e. $k=p$ prime, as follows: the orbit of L_1 under $\mathrm{O}(L_p,\det)$ consists of L_1 and $L_{1,p}$, where in general $L_{1,d} := \{\left(\begin{smallmatrix} d^{-1}a & c \\ c & db\end{smallmatrix}\right) \mid a,b,c\in\mathbf{Z}\}$. This is because L_1 must be mapped onto an isometric lattice contained in $L_p^{\sharp}$ and containing L_p. The isometry fixing L_p and mapping L_1 onto $L_{1,p}$ is the reflection by the vector $\mathrm{diag}(-1,p)\in L_p$, which can also be realized by extending the operation via 2×2-matrices to $p^{-\frac{1}{2}}\left(\begin{smallmatrix} 0 & 1 \\ p & 0\end{smallmatrix}\right)$. This settles the case $d(k)=1$. Now assume the statement proved for $\mathrm{O}(L_d,\det)$ for all proper divisors d of k. Let $k=pk'$ for some prime divisor p of k. Obviously the orbit of $L_{k'}$ under the action of $\mathrm{O}(L_{k'},\det)$ is of length $p+1$, as is the orbit under $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k'\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$. Hence, the stabilizer of L_k in $\mathrm{O}(L_{k'},\det)$ is an extension of S_k by an elementary Abelian 2-group of rank $d(k)-1=d(k')$. An argument similar to the one above shows that this stabilizer is of index at most 2 in $\mathrm{O}(L_k,\det)$. That it is of index exactly 2 can then be seen via the element of $\mathrm{O}(L_p,\det)$ with spinor norm $-p$. (In [Que96] the precise element is given, cf. also [Mac81].)

Note, the elementary Abelian 2-group $\mathrm{O}(L_k,\det)/S_k$ acts regularly on the set $\{L_{1,d}\mid d \text{ divides } k\}$. In terms of the affine building belonging to the p-adic completion of the group, all $L_{1,d}$ with $p\nmid d\mid k$ belong to one vertex of the attached tree and all other $L_{1,d}$ belong to a different vertex, which is not of the same type as the first vertex. Finally L_p, resp. all L_d with $p\mid d$, belong to the edge connecting the two vertices. □

Now Proposition 6.2 can be completed:

PROPOSITION 6.4. *Under the hypothesis and notation of Proposition 6.2 the index of* $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ *in* $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$ *is* $2^{1+a}\prod(p+1)$, *where* p *runs through all prime divisors of* $\frac{d}{st}$ *and* a *is at most equal to the number of prime divisors of* st. *Moreover,* $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))/\,\mathrm{Inn}(\mathrm{Bil}_\Lambda^+(L))$ *is an elementary* 2*-group of rank* a.

Proof. This is an immediate consequence of Proposition 6.2 and Lemma 6.3. □

The question arises, whether there are examples for which the minimal possible index of $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ in $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$ according to Proposition 6.4 is attained, i.e. $a = 0$ and $d = st$. This is already possible in the group case; cf. Example 2.2 (ii).

PROPOSITION 6.5. *For a prime number* p *let* $c(p) = p - 1$ *if* p *is odd and* $c(2) = 2$. *Then, for any sequence of prime numbers* $p_1 < p_2 < \cdots < p_l$, *there are examples with* $\dim_{\mathbf{Q}} \mathcal{V} = 2\prod_{i=1}^{l} c(p_i)$, *where* $\mathcal{A}$ *is an image of a finite group algebra and* $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$, *where* $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ *is of (minimal) index* 2 *in* $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$. *If* $p_i \equiv 3 \pmod 4$ *for all* i *with* $p_i \neq 2$, *then* L *can be chosen so that each* $\phi \in \mathrm{Bil}_\Lambda^+(L)$ *is* $c_0 q(\phi)$*-modular.*

Proof. First construct a finite $\mathbf{C}$-irreducible subgroup $G(p)$ of $\mathrm{GL}_{c(p)}(\mathbf{Q})$ as follows: for $p = 2$ take the automorphism group of the quadratic lattice (which is a dihedral group of order 8); for p odd take the Frobenius group of order $p(p-1)$ in its action on the permutation module factored by the fixed points, which is then identified with $\mathbf{Q}^{1\times c(p)}$. Take the span of $-I_{c(p)}$ with this group to obtain $G(p) \leq \mathrm{GL}_{c(p)}(\mathbf{Q})$ of order $2p(p-1)$. The $G(p)$-lattices in $\mathbf{Q}^{1\times c(p)}$ are described in [NeP95a] p. 29: up to multiples they come in a chain $L_0(p) \geq L_1(p) \geq \cdots \geq L_{c(p)} = pL_0(p) \geq \cdots$, where $L_i(p)$ is of index p^i in $L_0(p)$. There exists an element n in the normalizer of $G(p)$ in $\mathrm{GL}_{c(p)}(\mathbf{Q})$ mapping $L_i(p)$ onto $L_{i+c(p)/2}(p)$. Choosing $L = L_i(p) \oplus L_{i+c(p)/2}(p)$ and taking the $G(p)$-invariant symmetric bilinear forms for $\mathrm{Bil}_\Lambda^+(L)$ gives the desired result for the case $d = p$. The case $i = 0$ for $p = 2$, resp. $i = \frac{p-3}{4}$ for $p \equiv 3 \pmod 4$, gives modularity. The general case of composite d is obtained from the above by taking tensor products. □

One should note that in the above proof one gets modular lattices by choosing $L = L_i \oplus L_{p-i}$ without having the big $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L)$, if i is not chosen as above. The same holds for the composite case. By now it should be clear that the existence of outer automorphisms and modularity of the lattices are different phenomena.

To end up, some explicit examples of $*$-depth zero will be given, where $\mathrm{End}_\Lambda(L) \cong \left(\begin{smallmatrix} \mathbf{Z} & 3\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{smallmatrix}\right)$. One easily checks that the unit group is generated by a, b, c and that the outer automorphism is induced by d with

$$a := \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d := \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}.$$

Note that defining relations for the inner, resp. outer, automorphism group are provided by $\bar{a}^2$, $\bar{b}^2$, $\bar{c}^2$, $(\bar{a}\bar{b})^3$ and $\bar{b}^2$, $\bar{c}^2$, $\bar{d}^2$, $(\bar{c}\bar{d})^2$, $(\bar{d}\bar{b})^6$ respectively. The fundamental domains in the hyperbolic plane identified with $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(\mathcal{V})/\mathbf{R}_{>0}$, where $\mathbf{R}_{>0}$ acts by multiplication, are triangles with vertices P, C_1, C_2 in the first case, where C_1 and C_2 are cusps, and P, C_1, M in the second case. The angles can be read off from the presentation. According to Example 3.7 there are seven possibilities for the equivalence type of $\mathrm{Bil}_\Lambda(L)$, parametrized by the exponent matrices of $\mathrm{End}_\Lambda(L \oplus L^*)$ given there. Only in four cases can one have outer automorphisms.

EXAMPLE 6.6.

(i) Take the fourth possibility in the list of Example 3.7. Then $L = L_1 \oplus L_2$ with $L_1^\# = L_1$ and $L_2^\# = 3L_2$, where the reciprocal lattices are taken with respect to a generator ϕ_1 of $\mathrm{Bil}_\Lambda(L_1)$, and $L_2 \leq L_1$ is necessarily of index $3^{n/2}$ in L_1 with $n := \dim(L_1)$. (Note: n must be even.) Representing $\mathrm{Bil}_\Lambda^+(L)$ by Gram matrices one gets $\mathrm{Bil}_\Lambda^+(L) = \{\left(\begin{smallmatrix} \alpha F_1 & \gamma X \\ \gamma X^{tr} & \beta F_2 \end{smallmatrix}\right) \mid \alpha, \beta, \gamma \in \mathbf{Z}\}$, where F_1 and F_2 are unimodular (Gram matrices for L_1 and L_2) and $XF_1^{-1}X^{tr} = 3F_2$. Obviously one has no outer isomorphism if F_1 and F_2 are not equivalent. In this case $\mathrm{Bil}_\Lambda^+(L)$ is not modular, though ι is bijective, but it is not an equivalence. In any case, the vertices of the fundamental domain in this case are given by the $(\alpha, \beta, \gamma) \in \{(2,2,1), (1,0,0), (0,0,1)\}$ corresponding to P, C_1, C_2, the determinant is $(\alpha\beta - 3\gamma^2)^n$ and a nice realization of this setup is for $n = 12$, where one can find the 3-scaled version of the unimodular lattice D_{12}^+ as a sublattice of the standard lattice of index 3^6. Things can be so chosen that the 2-fold cover of the Mathieu group M_{12} acts. In $\mathrm{Bil}^+_{\Lambda,>0}(L)$ one has two orbits of primitive M_{12}-perfect lattices, one unimodular with minimum 2 and one of determinant 5^{12} with minimum 4. Obviously one can produce many more examples in higher dimensions. One can show that there is no realization of this situation for $n < 12$.

If one has an outer automorphism, there seems to be the possibility that $\mathrm{Bil}_{\Lambda}^{+}(L)$ is modular. The vertices of the fundamental domain in this case are given by the $(\alpha, \beta, \gamma) \in \{(2,2,1), (1,0,0), (1,1,0)\}$ corresponding to P, C_1, M. For the case $F_1 = F_2$ I have computed some examples: $F_1 = I_4$, E_8, Λ_{24} (Leech lattice). In the first case the vertex P represents the root lattice E_8, which is the only perfect lattice here. In the other two cases my choice of X (there might be more than one!) yielded a 6-modular form as the only perfect form; the coordinates were $(3,3,1)$, the minima were 6 and 12 respectively.

(ii) Take the eighth possibility in the list of Example 3.7. Then $L = L_1 \oplus L_2$ with $3L_1 < L_2 = 3L_1^{\#} < L_1 = 3L_2$, where the reciprocal lattices are taken with respect to a generator ϕ_1 of $\mathrm{Bil}_{\Lambda}(L_1)$.

Again representing $\mathrm{Bil}_{\Lambda}^{+}(L)$ by Gram matrices with respect to suitably chosen bases one gets $\mathrm{Bil}_{\Lambda}^{+}(L) = \{\left(\begin{smallmatrix} \alpha F & \gamma I_n \\ \gamma I_n & \beta \tilde{F} \end{smallmatrix}\right) \mid \alpha, \beta, \gamma \in \mathbf{Z}\}$, where F are the Gram matrices for (L_1, ϕ_1) and $\widetilde{F} = 3F^{-1}$. The determinant is $(3\alpha\beta - \gamma^2)^n$. Obviously one has an outer isomorphism if and only if F and $\widetilde{F}$ are $\mathbf{Z}$-equivalent, i.e. if (L_1, ϕ_1) is 3-modular. Many such examples, with and without outer automorphisms and also for other exponents different from 3 of $L_1^{\#}/L_1$, have been investigated in [Bav97], because in this case $\mathrm{Bil}_{\Lambda}^{-}(L)$ is spanned by unimodular symplectic forms. By Proposition 5.7 $\mathrm{Bil}_{\Lambda}^{+}(L)$ is modular. Here are some examples with outer automorphisms: $F = A_2$, $A_2 \otimes E_8$, K_{12} (the Coxeter-Todd lattice), and $[\pm S_6(3) \overset{2}{\square} C_3]_{26}$ of [Neb96b]; one gets one relative extremal lattice with coordinates $(\alpha, \beta, \gamma) = (1,1,1)$. They are 2-modular with minima 2, 4, 4, and 6 respectively. However, $F = [\mathrm{SL}_2(9) \underset{\infty,3}{\overset{2(3)}{\otimes}} \mathrm{SL}_2(9).2]_{16}$, which is also 3-modular with minimum 4 of dimension 16 (like $A_2 \otimes E_8$), yields the 11-modular form with minimum 12 and coordinates $(\alpha, \beta, \gamma) = (3,3,4)$ as extremal lattice. Finally, $F = N_{23}$ (the extremal 3-modular lattice of dimension 24 of [Neb95]; or [Neb98b], Theorem 5.1 for an alternative construction) yields a 23-modular lattice as extremal with minimum $24 = 4 \cdot 6$ and coordinates $(\alpha, \beta, \gamma) = (4,4,5)$. It would be interesting to investigate the density function on the fundamental domain theoretically.

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