6. BOUNDED EULER CLASS

of Γ since, once again, Γ commutes with θ.

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of Γ on $\mathbf{S}^1$. As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in $[x, \theta(x)[$ is contractible, by definition. This means that any compact interval in $\mathbf{S}^{1'}$ is contractible and this implies (STRONG EXPANSIVITY).

We have now proved that if (MINIMALITY) *and* (EXPANSIVITY) *are both satisfied, then the group* Γ *must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group Γ. If a group Γ acts equicontinuously, then its closure in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a compact group by Ascoli's theorem. We analyzed compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$ in 4.1 : they turned out to be abelian and conjugate to groups of rotations.

We have shown that if (MINIMALITY) *holds then* Γ *is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let Γ be any subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let us use the structure theorem 5.6–5.8. If Γ is minimal, we have already proved the theorem. If Γ has a finite orbit, there is a Γ-invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the Γ-action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that Γ contains a non abelian free subgroup unless the restriction of the action of Γ to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a Γ-invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

6. BOUNDED EULER CLASS

6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let Γ be any group. Let us consider the (semi)-simplicial set $E\Gamma$ whose vertices are the elements of Γ and for which n-simplices are all $(n+1)$-tuples of elements of Γ. The i^{th} face of the simplex $(\gamma_0, \dots, \gamma_k)$ is $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$ where the term γ_i is omitted. Note that the set $E\Gamma$ does not depend on the group structure of Γ.

As a matter of fact, $E\Gamma$ is contractible since it is the full simplex over the set Γ. However, there is a simplicial free action of Γ on $E\Gamma$ induced by left translations of Γ on itself. Hence once could think of the quotient $B\Gamma$ of $E\Gamma$ by this action as a space whose fundamental group is Γ and with vanishing higher homotopy groups. One would like to define the cohomology of the group Γ as the cohomology of this quotient space $B\Gamma$. We should be careful with $B\Gamma$ since it has only one vertex (a group acts transitively on itself !).

However, guided by this idea, it is natural to define a *k-cochain* of Γ with values in some abelian group A as a map $c\colon \Gamma^{k+1} \to A$ which is *homogeneous*, *i.e.* such that $c(\gamma\gamma_0, \gamma\gamma_1, \ldots, \gamma\gamma_k) = c(\gamma_0, \gamma_1, \ldots, \gamma_k)$ identically. The set of these cochains is an abelian group denoted by $C^k(\Gamma, A)$. We have a natural coboundary d_k from $C^k(\Gamma, A)$ to $C^{k+1}(\Gamma, A)$ defined by

$$d_k c(\gamma_0, \ldots, \gamma_{k+1}) = \sum_{i=0}^{k} (-1)^i c(\gamma_0, \ldots, \widehat{\gamma_i}, \ldots, \gamma_k).$$

Of course, we have $d_{k+1} \circ d_k = 0$ and we define the *cohomology group* $H^k(\Gamma, A)$ as being the quotient of cocycles (*i.e.* the kernel of d_k) by coboundaries (*i.e.* the image of d_{k-1}). If A is moreover a ring, then there is a natural cup product from $H^k(\Gamma, A) \times H^l(\Gamma, A)$ to $H^{k+l}(\Gamma, A)$. We refer to [11] for an excellent account of this theory of group cohomology. Note that for any homomorphism ϕ from a group Γ to another group Γ', there is an induced homomorphism $\phi^\star\colon H^k(\Gamma', A) \to H^k(\Gamma, A)$.

A homogeneous map $c\colon \Gamma^{k+1} \to A$ can be written in a unique way in the form $c(\gamma_0, \ldots, \gamma_k) = \bar{c}(\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, \ldots, \gamma_{k-1}^{-1}\gamma_k)$ for a unique function $\bar{c}\colon \Gamma^k \to A$. Conversely, given a map $\bar{c}$ there is a unique homogeneous map c satisfying this relation. One says that $\bar{c}$ is the inhomogeneous cochain associated to c. In other words, the space $C^k(\Gamma, A)$ is canonically isomorphic to the A-module of all maps $\Gamma^k \to A$.

In degree 1, a cochain is a homogeneous map $c\colon \Gamma^2 \to A$ and the corresponding inhomogeneous cochain is a map $\bar{c}\colon \Gamma \to A$. It is interesting to check that c is a cocycle if and only if $\bar{c}$ is a homomorphism. Moreover 0-cochains are constant maps from Γ to A and their coboundary is therefore 0. It follows that *for any group* Γ, *the cohomology* $H^1(\Gamma, A)$ *is identified with the set of homomorphisms from* Γ *to* A.

In degree 2, the interpretation is quite interesting. Consider a central extension of Γ by A :

$$0 \longrightarrow A \xrightarrow{i} \widetilde{\Gamma} \xrightarrow{p} \Gamma \longrightarrow 1.$$

This means that $\widetilde{\Gamma}$ contains a subgroup isomorphic to A contained in its center and that the quotient by this subgroup is isomorphic to Γ. Suppose that the projection p has a section s which is a homomorphism from Γ to $\widetilde{\Gamma}$ such that $p \circ s = Id_\Gamma$. Then it follows that $\widetilde{\Gamma}$ is isomorphic to the direct product $\Gamma \times A$ by the homomorphism sending (γ, a) to $s(\gamma)i(a)$. Hence, in order to measure the non triviality of an extension we try to find the "obstruction" to finding a section s. This is done in the following way. Choose a set theoretical section s from Γ to $\widetilde{\Gamma}$; this is possible since p is onto. If γ_1 and γ_2 are two elements of Γ, consider $\bar{c}(\gamma_1, \gamma_2) = s(\gamma_1\gamma_2)^{-1}s(\gamma_1)s(\gamma_2)$. This element projects on the identity element of Γ under p since p is a homomorphism; it is therefore an element of the image of i and can be identified with an element of A. This defines a map $\bar{c}\colon \Gamma^2 \to A$. Let $c\colon \Gamma^3 \to A$ be the associated homogeneous cochain. One checks that c is a cocycle. Of course, the section s is not unique but another choice s' has the form $s'(\gamma) = s(\gamma)i(u(\gamma))$ for some function $u\colon \Gamma \to A$. If one computes the cocycle c' associated to this new choice of a section s', one finds that $c' - c$ is the coboundary of the 1-cochain associated to the map u. It follows that the cohomology class of c in $H^2(\Gamma, A)$ is well defined, *i.e.* does not depend on the choice of a section. This cohomology class is called the *Euler class of the extension* under consideration.

It is not difficult to check the following properties of the Euler class.

1) Two central extensions $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of A by Γ are isomorphic by some isomorphism which is the identity on the central subgroup A and inducing the identity on the quotient Γ if and only if they have the same Euler class in $H^2(\Gamma, A)$.

2) Any class in $H^2(\Gamma, A)$ corresponds to a central extension.

In short, $H^2(\Gamma, A)$ *parametrizes isomorphism classes of central extensions of* A *by* Γ.

Before coming back to the dynamics of groups acting on the circle, let us consider a few simple examples.

If $\Gamma = \mathbf{Z}$, it is clear that every extension admits a section which is a homomorphism: it suffices to choose arbitrarily $s(1)$ in $p^{-1}(1)$ and to define $s(n) = s(1)^n$ for $n \in \mathbf{Z}$. Hence, if $\Gamma = \mathbf{Z}$ or more generally if Γ is a free group, we have $H^2(\Gamma, A) = 0$.

Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$. It has a presentation of the form

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle .$$

Now consider the group $\widetilde{\Gamma}_g$ defined by the presentation

$$\widetilde{\Gamma}_g = \big\langle z, a_1, b_1, \ldots, a_g, b_g \mid \\ a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \big\rangle\,.$$

The central subgroup A generated by z turns out to be infinite cyclic so that $\widetilde{\Gamma}_g$ defines a central extension of Γ_g by $\mathbf{Z}$, hence an Euler class in $H^2(\Gamma_g, \mathbf{Z})$. It is a fact that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic with $\mathbf{Z}$ and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus $g \geq 1$ has a contractible universal cover and that the cohomology of Γ_g can therefore be identified with the cohomology of the compact oriented surface of genus g (see [11] for more details).

6.2 The Euler class of a group action on the circle

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1\,.$$

The cohomology group $H^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ has been computed. It is isomorphic to $\mathbf{Z}$ and a generator is the Euler class of this central extension [50].

Consider now a homomorphism ϕ from some group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. Then, we can pull back the previous extension by ϕ. In other words, we consider the set of $(\gamma, \widetilde{f}) \in \Gamma \times \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi(\gamma) = p(\widetilde{f})$. This is a group $\widetilde{\Gamma}$ equipped with a canonical projection onto Γ whose kernel is isomorphic to $\mathbf{Z}$, *i.e.* $\widetilde{\Gamma}$ is a central extension of Γ by $\mathbf{Z}$. In case ϕ is injective, $\widetilde{\Gamma}$ is just the pre-image of $\phi(\Gamma)$ under p, which is the group of lifts of $\phi(\Gamma)$. The Euler class of this central extension of Γ is called *the Euler class of the homomorphism* ϕ and denoted by $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$. It is obviously a dynamical invariant in the sense that two conjugate homomorphisms ϕ_1 and ϕ_2 have the same Euler class in $H^2(\Gamma, \mathbf{Z})$. Note that it follows from the definition that $eu(\phi)$ is zero if and only if the homomorphism ϕ lifts to a homomorphism $\widetilde{\phi}\colon \Gamma \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi = p \circ \widetilde{\phi}$.

A few examples are in order. In the case of a single homeomorphism, *i.e.* when $\Gamma = \mathbf{Z}$, we saw that $H^2(\mathbf{Z}, \mathbf{Z}) = 0$. Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when Γ is free.

If Γ_g is the fundamental group of a closed oriented surface of genus $g \geq 1$, we know that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ so that the Euler class

$eu(\phi)$ in this case is an integer. In [51], Milnor gives an algorithm to compute this number. With the same notation as above, for each $1 \leq i \leq g$, choose lifts $\widetilde{a}_i$ and $\widetilde{b}_i$ of $\phi(a_i)$ and $\phi(b_i)$. Now compute the product of commutators $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\ldots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. Since this homeomorphism is a lift of the identity, it is an integral translation. This amplitude of this translation does not depend on the choices made and is the Euler number $eu(\phi)$.

As an explicit example, also computed by Milnor, recall that any closed orientable surface of genus $g > 1$ can be endowed with a riemannian metric of constant negative curvature. Recall also that the Poincaré upper half space $\mathcal{H}$ can be equipped with a metric of curvature -1 whose group of orientation preserving isometries is precisely $\mathrm{PSL}(2,\mathbf{R})$. Moreover, any complete simply connected riemannian surface of curvature -1 is isometric to $\mathcal{H}$. Hence there are embeddings ϕ of the fundamental group Γ_g of a closed oriented surface of genus $g > 1$ in $\mathrm{PSL}(2,\mathbf{R})$ such that the corresponding action of Γ_g on $\mathcal{H}$ is free, proper and cocompact. Since we know that $\mathrm{PSL}(2,\mathbf{R})$ is a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we can compute the corresponding Euler number $eu(\phi)$. The result of the computation is $2g-2$. Note that each element of $\phi(\Gamma_g)$ is hyperbolic since the action is free and cocompact so that the rotation number of every element of $\phi(\Gamma_g)$ is 0. So we are in a situation in which the topological invariant $eu(\phi)$ is not 0 but the rotation number invariants are trivial; a situation different from the case where $\Gamma = \mathbf{Z}$.

6.3 Bounded cohomology and the Milnor-Wood inequality

It was observed very early that the Euler class of a homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ cannot be arbitrary. Milnor and Wood proved the following [51, 71].

Theorem 6.1 (Milnor-Wood). *Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$ and $\phi\colon \Gamma_g \to \mathrm{Homeo}_+(\mathbf{S}^1)$ be any homomorphism. Then the Euler number satisfies $|eu(\phi)| \leq 2g-2$.*

Proof. We shall not give a complete proof since this result will follow from later considerations but we prove a weaker version. Keeping the previous notation, we know that $eu(\phi)$ is the translation number of the homeomorphism $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\ldots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. We also know that the translation number function τ is a quasi-homomorphism, *i.e.* there is some inequality of the form $\left|\tau(\widetilde{f}_1\widetilde{f}_2) - \tau(\widetilde{f}_1) - \tau(\widetilde{f}_2)\right| \leq D$ for some D. We also know that $\tau\left(\widetilde{f}^{-1}\right) = -\tau(\widetilde{f})$. So, if we evaluate τ on this element, we get a bound of the form $|eu(\phi)| \leq (4g-1)D$. This is not quite the bound given in the theorem but this explains the idea of the proof: to get the exact bound, one should be a little bit more clever! □

In [17], Eisenbud, Hirsch and Neumann gave a much more precise result that we would like to mention here. If $\widetilde{f}$ is an element of $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$, define $\underline{m}(\widetilde{f}) = \inf(\widetilde{f}(x)-x)$ and $\overline{m}(\widetilde{f}) = \sup(\widetilde{f}(x)-x)$. Note that $\underline{m}(\widetilde{f}) \leq \tau(\widetilde{f}) \leq \overline{m}(\widetilde{f})$ and $0 \leq \overline{m}(\widetilde{f}) - \underline{m}(\widetilde{f}) < 1$.

THEOREM 6.2 (Eisenbud, Hirsch, Neumann). *An element $\widetilde{f}$ of the group $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ can be written as a product of $g \geq 1$ commutators if and only if $\underline{m}(\widetilde{f}) < 2g-1$ and $1-2g < \overline{m}(\widetilde{f})$.*

Any element of $\text{Homeo}_+(\mathbf{S}^1)$ has at least one lift $\widetilde{f}$ in $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ such that $-1 < \underline{m}(\widetilde{f}) \leq \overline{m}(\widetilde{f}) < 1$ so that it can be written as one commutator. It follows that every element of $\text{Homeo}_+(\mathbf{S}^1)$ can be written as a commutator. We mentioned this fact earlier.

In [25], we put these inequalities in the context of bounded cohomology, which was introduced by Gromov (see [30] for many geometrical motivations). Consider again an abstract group Γ and let $A = \mathbf{Z}$ or $\mathbf{R}$. Then define a *bounded* k-cochain as a bounded homogeneous map from Γ^{k+1} to A. This defines a sub A-module of $C^k(\Gamma, A)$ denoted by $C^k_b(\Gamma, A)$. It is clear that the coboundary d_k of a bounded k-cochain is a bounded $(k+1)$-cochain so that we can define the cohomology of this new differential complex, that is called the *bounded cohomology of* Γ *with coefficients in* A and denoted by $H^k_b(\Gamma, A)$. We have obvious maps from $H^k_b(\Gamma, A)$ to $H^k(\Gamma, A)$ obtained by "forgetting" that a cocycle is bounded. In general these maps are neither injective nor surjective. See [35, 36] for a detailed algebraic background on this cohomology.

The degree 1 case is trivial. A cocycle is given by a bounded homomorphism from Γ to A and is therefore trivial. Hence $H^1_b(\Gamma, A) = 0$ for any group Γ.

The degree 2 case is the most interesting for us. Let us look first at $H^2_b(\mathbf{Z}, \mathbf{R})$. Consider a bounded 2-cocycle c on $\mathbf{Z}$ with values in $\mathbf{R}$. Since we know that $H^2(\mathbf{Z}, \mathbf{R}) = 0$, we know that c is the coboundary of a 1-cochain of the form $u(n_1, n_2) = \overline{u}(n_1 - n_2)$ for some function $\overline{u}\colon \mathbf{Z} \to \mathbf{R}$. The fact that c is bounded means precisely that $\overline{u}$ is a quasi-homomorphism from $\mathbf{Z}$ to $\mathbf{R}$. We know that this implies the existence of a real number τ such that $\overline{u}(n) - n\tau$ is bounded. Now, if we define $\overline{v}(n) = \overline{u}(n) - n\tau$, then the coboundary of the *bounded* 1-cochain $v(n_1, n_2) = \overline{v}(n_1 - n_2)$ is c. We have shown that $H^2_b(\mathbf{Z}, \mathbf{R}) = 0$.

For a general group Γ, let us define $QM(\Gamma)$ as being the vector space of quasi-homomorphisms from Γ to $\mathbf{R}$. Say that a quasi-homomorphism is trivial if it differs from some homomorphism by a bounded amount. It follows from the definitions and the previous argument that the kernel of the map from $H^2_b(\Gamma, \mathbf{R})$ to $H^2(\Gamma, \mathbf{R})$ is precisely the quotient of $QM(\Gamma)$ by the subspace of trivial quasi-homomorphisms. This gives some intuition about the group $H^2_b(\Gamma, \mathbf{R})$.

Let us compute now some examples with coefficients in $\mathbf{Z}$. Start with $H^2_b(\mathbf{Z}, \mathbf{Z})$. Let c be a bounded integral 2-cocycle. We know that it is the coboundary of a 1-cochain of the form $u(n_1, n_2) = \overline{u}(n_1 - n_2)$ for some function $\overline{u}\colon \mathbf{Z} \to \mathbf{Z}$. Again, we know that there is a real number τ such that $\overline{u}(n) - n\tau$ is bounded but if we define $\overline{v}(n) = \overline{u}(n) - n\tau$ the 1-cochain v is not integral unless τ is an integer! For each real number τ, define c_τ to be the coboundary of the integral 1-cochain $v_\tau(n_1, n_2) = [(n_1 - n_2)\tau]$ where [] denotes the integral part of a real number. It is clear that c_τ is bounded (by 1) and our previous computations show that every bounded integral 2-cocycle in $\mathbf{Z}$ is cohomologous to some c_τ for some τ. Moreover, it is clear that c_{τ_1} and c_{τ_2} define the same element in $H^2_b(\mathbf{Z}, \mathbf{Z})$ if and only if $\tau_1 - \tau_2$ is an integer. Summing up, we showed that $H^2_b(\mathbf{Z}, \mathbf{Z})$ is isomorphic to $\mathbf{R}/\mathbf{Z}$. We hope that the reader will recognize that the rotation number is showing up...

As a matter of fact, the argument that we presented is more general and shows immediately that for any group Γ, the kernel of the map from $H^2_b(\Gamma, \mathbf{Z})$ to $H^2_b(\Gamma, \mathbf{R})$ is precisely the quotient $H^1(\Gamma, \mathbf{R})/H^1(\Gamma, \mathbf{Z})$. (Recall that $H^1(\Gamma, A)$ is the set of homomorphisms from Γ to A.)

We now come to the construction of an invariant of a group action on the circle that combines the rotation numbers and the Euler class. Let us look again at the central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \longrightarrow \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1$$

and let us try to find some 2-cocycle representing its Euler class (see also [38]). We know that we should choose a set theoretical section s to p. It turns out that there is a natural choice of such a section. Indeed, let $f \in \mathrm{Homeo}_+(\mathbf{S}^1)$, then among the elements in $p^{-1}(f) \in \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$, there is only one, denoted by $\sigma(f)$, which is such that $\sigma(f)(0)$ lies in the interval $[0,1[\subset \mathbf{R}$. This σ will be our preferred section. Let us try to evaluate the associated 2-cocycle c on $\mathrm{Homeo}_+(\mathbf{S}^1)$. By definition the associated inhomogeneous cocycle $\overline{c}$ is:

$$\overline{c}(f_1, f_2) = \sigma(f_1 f_2)^{-1} \sigma(f_1) \sigma(f_2).$$

The main (easy) observation is that the cocycle c is bounded. More precisely:

LEMMA 6.3. *The 2-cocycle c takes only the two values 0 and 1.*

Proof. By definition $\sigma(f_2)(0)$ is in $[0,1[$. It follows that $\sigma(f_1)(\sigma(f_2)(0))$ is in the interval $[\sigma(f_1)(0), \sigma(f_1)(0)+1[$ which is contained in $[0,2[$. We know that $\sigma(f_1f_2)$ and $\sigma(f_1)\sigma(f_2)$ are lifts of the same element f_1f_2 and that $\sigma(f_1f_2)(0)$ is in $[0,1[$. It follows that $\sigma(f_1f_2)^{-1}\sigma(f_1)\sigma(f_2)$ is the translation by 0 or 1. □

Hence, for this choice of section σ, the associated 2-cocycle c is bounded and integral. Thus, we have defined an element of $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ that we call the *bounded Euler class*. It may seem that the definition depends on the choice of the origin 0 on the line but the reader will easily check that a modification of the origin would change the section σ by a bounded amount so that the bounded integral cohomology class is indeed well defined. If we have a homomorphism ϕ from a group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ we can pull back this bounded Euler class. We get an element in $H_b^2(\Gamma, \mathbf{Z})$ that we still denote by $eu(\phi)$ and that we call the bounded Euler class of the homomorphism ϕ. In case $\Gamma = \mathbf{Z}$, it should now be clear that the corresponding bounded Euler class in $H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ is exactly the rotation number of the homeomorphism $\phi(1)$. Hence we have proved the following:

THEOREM 6.4 ([25]). *There is a class eu in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ such that:*

1) *For every homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ the image of $\phi^*(eu) \in H_b^2(\Gamma, \mathbf{Z})$ in $H^2(\Gamma, \mathbf{Z})$ under the canonical map is the Euler class.*

2) *If $\phi\colon \mathbf{Z} \to \mathrm{Homeo}_+(\mathbf{S}^1)$ is a homomorphism then $\phi^*(eu) \in H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ is the rotation number of $\phi(1)$.*

3) *$\phi^*(eu)$ is a topological invariant, i.e. if ϕ_1 and ϕ_2 are two homomorphisms from Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ which are conjugate by an orientation preserving homeomorphism, then $\phi_1^*(eu) = \phi_2^*(eu)$ in $H_b^2(\Gamma, \mathbf{Z})$.*

In other words, the bounded Euler class is a topological invariant which combines the Euler class and the rotation number.

We now show that this new invariant for a group action is as powerful as the rotation number was for a single homeomorphism. Let us begin by the most interesting case.

THEOREM 6.5 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that all orbits are dense on the circle. Assume that the bounded Euler classes are equal: $\phi_1^*(eu) = \phi_2^*(eu)$. Then ϕ_1 and ϕ_2 are conjugate by an orientation preserving homeomorphism.*

Proof. This is very similar to the corresponding statement for rotation numbers: compare with the proof of 5.9. Since $\phi_1^*(eu) = \phi_2^*(eu)$ then in particular the Euler classes in $H^2(\Gamma, \mathbf{Z})$ are equal, which means that ϕ_1 and ϕ_2 define the same central extension $\widetilde{\Gamma}$. In other words, there is a central extension $0 \to \mathbf{Z} \to \widetilde{\Gamma} \to \Gamma \to 1$ and homomorphisms $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ from $\widetilde{\Gamma}$ to $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\widetilde{\phi}_1$ and $\widetilde{\phi}_1$ map the generator 1 of $\mathbf{Z}$ on the translation by 1 and such that the induced homomorphisms from $\widetilde{\Gamma}/\mathbf{Z} \simeq \Gamma$ to $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)/\mathbf{Z} \simeq \mathrm{Homeo}_+(\mathbf{S}^1)$ are ϕ_1 and ϕ_2. The assumption that the bounded classes agree means in fact that we can choose those homomorphisms in such a way that for each x in $\mathbf{R}$, the points $\widetilde{\phi}_1(\widetilde{\gamma})\widetilde{\phi}_2(\widetilde{\gamma})^{-1}(x)$ are bounded independently of $\widetilde{\gamma}$ in $\widetilde{\Gamma}$. We now define $\widetilde{h}(x)$ to be the upper bound of this bounded set. This map $\widetilde{h}$ is increasing, commutes with integral translations, and conjugates $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$. The jump and plateau sets of $\widetilde{h}$ are open sets invariant under $\widetilde{\phi}_1(\widetilde{\Gamma})$ and $\widetilde{\phi}_2(\widetilde{\Gamma})$ respectively. By our assumption these open sets are empty so that $\widetilde{h}$ is a homeomorphism which induces a conjugacy between ϕ_1 and ϕ_2. For more details, see [25]. □

In case the group $\phi(\Gamma)$ does not have all its orbits dense, we saw in 5.6 that there are two possibilities: $\phi(\Gamma)$ can have a finite orbit or $\phi(\Gamma)$ can have an exceptional minimal set. In the second case, we also saw that there is a canonical way of "collapsing" the connected components of the complement of the exceptional minimal set to construct another homomorphism $\overline{\phi}$ which has all its orbits dense: this is the associated "minimal" homomorphism (see 5.8).

Suppose now that $\phi(\Gamma)$ has a finite orbit consisting of k elements. Then, every element of $\phi(\Gamma)$ must permute these k points cyclically so that we get a homomorphism $r \colon \Gamma \to \mathbf{Z}/k\mathbf{Z}$. It is clear that two finite orbits of $\phi(\Gamma)$ have the same number of points and define the same r: we call this r the *cyclic structure of the finite orbits.* Conversely, consider a homomorphism $r \colon \Gamma \to \mathbf{Z}/k\mathbf{Z}$ and the corresponding action on the circle by rotations of order k. The bounded Euler class of this action is an element of $H_b^2(\Gamma, \mathbf{Z})$: we call these elements the *rational elements* in $H_b^2(\Gamma, \mathbf{Z})$. It is not difficult to see that an element in $H_b^2(\Gamma, \mathbf{Z})$ is rational if and only if its pull-back on some finite index subgroup is trivial.

Now, we can state the general result which is the exact analogue of what has been done in 5.9 for the rotation number. We don't give the proof: it can be found in [25] (in a slightly different terminology and with small mistakes...), but the reader should now be in a condition to fill in the missing details by himself.

THEOREM 6.6 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Assume that the bounded Euler classes $\phi_1^*(eu) = \phi_2^*(eu)$ are equal to the same class c in $H_b^2(\Gamma, \mathbf{Z})$.*

1) *If c is a rational class, then $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits with the same cyclic structure.*

2) *If c is not rational, then the associated minimal homomorphisms $\overline{\phi}_1$ and $\overline{\phi}_2$ are conjugate.*

Conversely, if $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits of the same cyclic structure or if they have no finite orbit and their associated minimal homomorphisms are conjugate (by an orientation preserving homeomorphism), then they have the same bounded Euler class.

Note in particular that the bounded Euler class of an action vanishes if and only if there is a point on the circle which is fixed by all the elements of the group.

6.4 EXPLICIT BOUNDS ON THE EULER CLASS

Since we know that the bounded Euler class of an action contains almost all the topological information, it is very natural to try to determine the part of $H_b^2(\Gamma, \mathbf{Z})$ which corresponds to the bounded Euler classes of all actions of Γ on the circle. In the case $\Gamma = \mathbf{Z}$, we know that $H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ and that every class corresponds to an action (by rotations). However, in the case where Γ is the fundamental group of a closed oriented surface of genus $g \geq 1$, the Milnor-Wood inequality shows that even the usual Euler class in $H^2(\Gamma, \mathbf{Z}) = \mathbf{Z}$ has to satisfy some inequality.

Given a bounded cochain c in $C_b^k(\Gamma, \mathbf{R})$, we define its norm $\|c\|$ as the supremum of the absolute value of $c(\gamma_0, \ldots, \gamma_k)$. Then we define the "norm" of a bounded cohomology class with real coefficients as the infimum of the norms of cocycles that represent it. We should be aware of the fact that this norm is not really a norm but is merely a semi-norm: a non zero class might have zero norm... Consider the case of the bounded Euler class, seen in the real bounded cohomology.

THEOREM 6.7. *The image of the bounded Euler class eu in the real bounded cohomology* $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ *has norm* $1/2$.

Proof. This is the abstract version of the Milnor-Wood inequality. Note that a constant 2-cocycle is the coboundary of a constant 1-cochain. We found a representative of the Euler class taking only two values 0 and 1. If we subtract from this cocycle the constant cocycle taking the value $1/2$, we get a cohomologous bounded (real) cocycle taking values $\pm 1/2$. This shows that the norm of the image of eu in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ is at most $1/2$. The opposite inequality follows from Milnor's computation of the Euler number for an embedding of the fundamental group Γ_g of a closed oriented surface as a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbf{R})$ that we mentioned in 6.1. If the norm were strictly less than $1/2$, then this number would be strictly less than $2g-2$. See [25] for more explanations. □

6.5 ACTIONS ON THE REAL LINE AND ORDERINGS

Our main concern is to study actions on the circle but there is a preliminary question which deals with actions on the line. Of course, if a group acts on the line, we can always add a point at infinity to produce an action on the circle (with a common fixed point). In other words studying actions on the line is equivalent to studying actions on the circle with vanishing bounded Euler class. This is the reason why we begin by general remarks on groups acting on the line.

Observe first that the dynamics of a single orientation preserving homeomorphism h of $\mathbf{R}$ are very easy to describe. Let $F = Fix(h)$ be the set of fixed points. Each interval of the complement of F is h-invariant and the action of h on this interval is conjugate to a translation (positive or negative, according to the sign of $h(x) - x$ on this interval).

We say that a group Γ is *left orderable* if there exists a *total* ordering $\preceq$ on Γ which is invariant under left translations (*i.e.* $\gamma_1 \preceq \gamma_2$ implies $\gamma\gamma_1 \preceq \gamma\gamma_2$). We write $\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. An obvious necessary condition for a group to be left orderable is that it be torsion free (*i.e.* there is no non trivial element of finite order).

The following theorem is well known but we weren't able to find its origin in the literature.

THEOREM 6.8. *Let* Γ *be a countable group. Then the following are equivalent:*

1) Γ *acts faithfully on the real line by orientation preserving homeomorphisms.*

2) Γ *is left orderable.*

Proof. Suppose that Γ acts faithfully on the line by orientation preserving homeomorphisms, *i.e.* that there exists an injective homomorphism ϕ from Γ into the group $\mathrm{Homeo}_+(\mathbf{R})$ of orientation preserving homeomorphisms of the real line. Assume first that there is a point x_0 in $\mathbf{R}$ with trivial stabilizer. Then we can define a left invariant total ordering by defining $\gamma_1 \preceq \gamma_2$ if $\phi(\gamma_1)(x_0) \leq \phi(\gamma_2)(x_0)$. If there is no such point x_0, choose a sequence of points $(x_i)_{i\in\mathbf{N}}$ which is dense in the line. Now define $\gamma_1 \preceq \gamma_2$ if $\gamma_1 = \gamma_2$ or if the first i for which $\phi(\gamma_1)(x_i) \neq \phi(\gamma_2)(x_i)$ is such that $\phi(\gamma_1)(x_i) < \phi(\gamma_2)(x_i)$. This defines a left invariant total order on Γ.

Conversely, let $\preceq$ be a left invariant total order on the countable group Γ. Enumerate the elements of Γ, *i.e.*, choose a bijection $i \in \mathbf{N} \mapsto \gamma_i \in \Gamma$. We are going to construct inductively an increasing injection v of $(\Gamma, \preceq)$ in $(\mathbf{R}, \leq)$. Define $v(\gamma_0)$ arbitrarily and suppose that $v(\gamma_0), \ldots, v(\gamma_i)$ have been defined. If γ_{i+1} is smaller (resp. bigger) than all $\gamma_0, \ldots, \gamma_i$ then define $v(\gamma_{i+1})$ as any real number smaller (resp. bigger) than $\min(v(\gamma_0), \ldots, v(\gamma_i)) - 1$ (resp. $\max(v(\gamma_0), \ldots, v(\gamma_i)) + 1$). Otherwise, there is a pair of integers $0 \leq \alpha, \beta \leq i$ such that $\gamma_\alpha \prec \gamma_{i+1} \prec \gamma_\beta$ and such that there is no γ_j $(0 \leq j \leq i)$ between γ_α and γ_β. Then we define $v(\gamma_{i+1})$ as $(v(\gamma_\alpha) + v(\gamma_\beta))/2$. Let $\bar{X} \subset \mathbf{R}$ be the closure of $v(\Gamma)$.

By our construction, it is easy to verify that $\bar{X}$ is unbounded and that any connected component $]a, b[$ of the complement of $\bar{X}$ is such that a and b are in $v(\Gamma)$. The group Γ acts on itself by left translations so that every element γ of Γ induces an increasing bijection $\phi(\gamma)$ of $v(\Gamma)$. We claim that $\phi(\gamma)$ extends continuously to $\bar{X}$. Otherwise, there would exist a point $x = \lim_n v(\gamma_{i_n}) = \lim_m v(\gamma_{i_m})$ for an increasing sequence of elements γ_{i_n} and a decreasing sequence γ_{i_m} and such that $\lim_n v(\gamma\gamma_{i_n}) < \lim_m v(\gamma\gamma_{i_m})$. Then $a = \lim_n v(\gamma\gamma_{i_n})$ and $b = \lim_m v(\gamma\gamma_{i_m})$ would be the endpoints of some connected component of the complement of $\bar{X}$. By our previous observation, a and b would be the image by v of two distinct elements of Γ. On multiplying these two elements on the left by γ^{-1}, this would produce two distinct elements γ_- and γ_+ such that $v(\gamma_{i_n}) \leq v(\gamma_-) < v(\gamma_+) \leq v(\gamma_{i_m})$ and this contradicts the fact that the two sequences have the same limit x.

Therefore we have produced a homeomorphism $\phi(\gamma)$ of $\bar{X}$. We now extend $\phi(\gamma)$ to the whole line $\mathbf{R}$ in such a way that $\phi(\gamma)$ is affine on each interval of the complement of $\bar{X}$. It is now clear that ϕ is an injective homomorphism from Γ to the group of orientation preserving homeomorphisms of the real line. $\square$

Theorem 6.8 produces many examples of actions on the real line. For instance, suppose Γ is a countable group containing a nested sequence of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_i \supset \ldots$ (finite or infinite) such that the intersection of this family reduces to the trivial element and that each Γ_i is a normal subgroup in the previous one Γ_{i-1}. Assume that each quotient $Q_i = \Gamma_i/\Gamma_{i-1}$ is left orderable and denote by $\preceq_i$ such a left order on Q_i. Let us construct a left order $\preceq$ on Γ. Consider two distinct elements γ, γ' in Γ and let i be the first integer such that $\gamma\gamma'^{-1}$ is not in Γ_i. Then $\gamma^{-1}\gamma'$ is in Γ_{i-1} and determines an element $[\gamma^{-1}\gamma']$ of Q_i. Then define $\gamma \preceq \gamma'$ if $[\gamma^{-1}\gamma'] \preceq_i 1$. This is a left invariant total order on Γ.

As an example, note that a countable torsion free abelian group A embeds in the tensor product $A \otimes \mathbf{Q}$ which is a $\mathbf{Q}$-vector space whose dimension is at most countable and therefore embeds in $\mathbf{R}$. Hence, countable torsion free abelian groups are orderable. Let us say that a group Γ is solvable (resp. residually solvable) if there is a finite (resp. infinite) decreasing sequence of subgroups as in the previous paragraph such that the quotient groups Q_i are abelian. We have now proved:

PROPOSITION 6.9. *Let Γ be a countable group which is (residually) solvable with torsion free abelian quotients. Then Γ acts faithfully on the real line by orientation preserving homeomorphisms.*

There are many examples of such groups: free groups or fundamental groups of closed orientable surfaces for instance have these properties [46]. Observe that the left orderings that we produced by the previous argument are in fact left *and right* invariant orderings. If we go back to the proof of Theorem 6.8 we can check that for bi-invariant ordered groups, the actions on the line $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{R})$ produced by the proof are very peculiar: they are such that for every non trivial $\gamma \in \Gamma$, we have either $\phi(\gamma)(x) \leq x$ for all $x \in \mathbf{R}$ or $\phi(\gamma)(x) \geq x$ for all x. In other words the graphs of $\phi(\gamma)$ don't cross the diagonal. However, there will be elements whose graphs touch the diagonals, unless of course the action is free, which is almost never the case because of the following well known theorem of Hölder.

THEOREM 6.10 (Hölder). *If a group acts freely on the real line by homeomorphisms, it is abelian. More precisely, such a group embeds as a subgroup of* $\mathbf{R}$ *and the action is semi-conjugate to a group of translations. In the same way, a group acting freely on the circle is abelian, embeds in* $\mathrm{SO}(2)$, *and is semi-conjugate to a group of rotations.*

Proof. Let $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{R})$ be a homomorphism such that for all γ different from the identity the homeomorphism $\phi(\gamma)$ has no fixed point. If γ, γ' are elements of Γ, write $\gamma \preceq \gamma'$ if $\phi(\gamma)(0) \leq \phi(\gamma')(0)$ (which implies $\phi(\gamma)(x) \leq \phi(\gamma')(x)$ for all x since the action is free). This defines a left and right invariant ordering $\preceq$ which is archimedean, *i.e.* such that for any pair of non trivial elements γ, γ' for which $id \prec \gamma$ and $id \prec \gamma'$, there is a positive integer n such that $\gamma' \prec \gamma^n$. Indeed, the sequence $\phi(\gamma)^n(0)$ is increasing and has to tend to ∞ since otherwise its limit would be a fixed point of $\phi(\gamma)$; hence for n sufficiently large we have $\phi(\gamma')(0) \leq \phi(\gamma^n)(0)$.

Then we show that any group Γ equipped with a bi-invariant total archimedean ordering embeds in $\mathbf{R}$ and is therefore abelian. Fix a non trivial element γ_0 such that $id \prec \gamma_0$ and for each $\gamma \in \Gamma$, define $\Phi(\gamma)$ as the smallest integer $k \in \mathbf{Z}$ such that $\gamma \preceq \gamma_0^k$. We have

$$\gamma_0^{\Phi(\gamma)-1} \prec \gamma \preceq \gamma_0^{\Phi(\gamma)}.$$

This defines a map $\Phi\colon \Gamma \to \mathbf{Z}$ which satisfies

$$\Phi(\gamma) + \Phi(\gamma') - 1 < \Phi(\gamma\gamma') \leq \Phi(\gamma) + \Phi(\gamma')$$

so that Φ is a quasi-homomorphism. As we have already observed, $\phi(\gamma) = \lim_{n\to\infty} \Phi(\gamma^n)/n$ exists and defines a quasi-homomorphism $\phi\colon \Gamma \to \mathbf{R}$ which is homogeneous (*i.e.* $\phi(\gamma^n) = n\phi(\gamma)$) and which is increasing (*i.e.* $\gamma \preceq \gamma'$ implies $\phi(\gamma) \leq \phi(\gamma')$). Note that $\phi(\gamma_0) = 1$.

We claim that ϕ is a group homomorphism. Indeed, consider two elements γ, γ' in Γ and assume for instance that $\gamma\gamma' \preceq \gamma'\gamma$. It follows easily by induction that for every positive integer n, we have $\gamma^n\gamma'^n \preceq (\gamma\gamma')^n \preceq \gamma'^n\gamma^n$. Evaluating Φ on this inequality, we get

$$\Phi(\gamma^n) + \Phi(\gamma'^n) - 1 \leq \Phi((\gamma\gamma')^n) \leq \Phi(\gamma^n) + \Phi(\gamma'^n).$$

Dividing by n and taking the limit, we obtain

$$\phi(\gamma) + \phi(\gamma') \leq \phi(\gamma\gamma') \leq \phi(\gamma) + \phi(\gamma')$$

so that ϕ is indeed a homomorphism.

We still have to show that ϕ is injective. For any γ such that $id \prec \gamma$ we know, since the ordering is archimedean, that there is some positive integer k

such that $\gamma_0 \preceq \gamma^k$. It follows that $1 \leq k\phi(\gamma)$ so that $\phi(\gamma)$ is non trivial. This proves the injectivity of ϕ.

Observe that the non decreasing embedding ϕ of Γ in $\mathbf{R}$ is unique up to a multiplicative constant. Indeed, if ϕ' is another one, we have by definition $(\Phi(\gamma^n)-1)\phi'(\gamma_0) \leq \phi'(\gamma^n) \leq \Phi(\gamma^n)\phi'(\gamma_0)$. Dividing by n and taking the limit, we get $\phi' = \phi'(\gamma_0).\phi$.

We now show that the action of Γ is semi-conjugate to a group of translations. If Γ is isomorphic to $\mathbf{Z}$, it acts freely and properly on the line so that it is indeed conjugate to the group of integral translations. Otherwise, $\phi(\Gamma)$ is dense in $\mathbf{R}$. Let x be any point in $\mathbf{R}$ and define

$$h(x) = \sup\{\phi(\gamma) \in \mathbf{R} \mid \gamma(0) \leq x\}.$$

Clearly, h is non decreasing and satisfies $h(\gamma(x)) = h(x) + \phi(\gamma)$ identically. The continuity of h is easy and follows from the density of the group $\phi(\Gamma)$: if h were not continuous, the interior of $\mathbf{R} \setminus h(\mathbf{R})$ would be a non empty open set invariant by all translations in $\phi(\Gamma)$.

The proof for groups acting on the circle follows easily: if Γ is a group acting freely on the circle, its inverse image in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ acts freely on the line. □

The following is an elementary corollary of the previous theorem.

PROPOSITION 6.11. *Let Γ be a torsion group (i.e. such that every element in Γ has finite order). Then any homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has abelian image.*

Proof. We know the structure of elements of finite order of $\mathrm{Homeo}_+(\mathbf{S}^1)$: they are conjugate to rotations of finite order. It follows that an element having a fixed point and of finite order in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is the identity. In other words, a torsion group acting faithfully on the circle acts freely. The result follows from 6.10. □

There is another very interesting example of a group which admits a left and right invariant total ordering: the group $\mathrm{PL}_+([0,1])$ of orientation preserving piecewise linear homeomorphisms of the interval $[0,1]$. Indeed, let γ, γ' be two distinct elements of $\mathrm{PL}_+([0,1])$ and consider the largest real number $x \in [0,1]$ such that γ and γ' coincide on the interval $[0,x]$. Then for $\epsilon > 0$ small enough, we have either $\gamma(t) < \gamma'(t)$ for $t \in \,]x, x+\epsilon]$ or $\gamma(t) > \gamma'(t)$ for $t \in \,]x, x+\epsilon]$. Say that $\gamma \prec \gamma'$ in the first case and $\gamma' \prec \gamma$ in the second case. This defines a total ordering on $PL_+([0,1])$ and it is clearly left and right invariant. We can induce this ordering on countable subgroups of $PL_+([0,1])$, for instance the subgroup of elements with rational slopes and apply the general construction that we described above. We get an action of this rational group on the line which is very different from the given action of $PL_+([0,1])$ on $]0,1[$: the corresponding graphs don't cross the diagonal.

Remark that an affine bijection of the line $x \mapsto ax+b$ has at most one fixed point (if it is not the identity). Solodov proved that this property essentially characterizes groups of affine transformations.

THEOREM 6.12 (Solodov). *Let Γ be a non abelian subgroup of* $\mathrm{Homeo}_+(\mathbf{R})$ *such that every element (different from the identity) has at most one fixed point. Then Γ is isomorphic to a subgroup of the affine group* $\mathrm{Aff}_+(\mathbf{R})$ *of the real line, and the action of Γ on the line is semi-conjugate to the corresponding affine action.*

Solodov did not publish a proof but mentions his result in [62] and explained it to the author of these notes in 1991. Later T. Barbot needed this theorem for his study of Anosov flows and published a proof in [3]. More recently, N. Kovačević published an independent proof in [43]. See also the recent preprint [20] for a detailed proof.

Proof. Let Γ be a subgroup of $\mathrm{Homeo}_+(\mathbf{R})$ such that every element (different from the identity) has at most one fixed point. If no non trivial element has a fixed point, Hölder's Theorem 6.10 implies that Γ is abelian (and that the action is semi-conjugate to a group of translations). If there is a point x which is fixed by the full group Γ, then one can restrict the action to the two components of $\mathbf{R} \setminus \{x\}$ on which we can use Hölder's theorem again: this would imply that Γ is abelian.

We claim that Γ contains an element γ with a repulsive fixed point x, i.e. such that $\gamma(y) > y$ for every $y > x$ and $\gamma(y) < y$ for every $y < x$. Indeed choose some non trivial γ_0 in Γ fixing some x_0. If x_0 is not repulsive for γ_0 and for γ_0^{-1}, this means that x_0 is a parabolic fixed point, *i.e.* replacing γ_0 by its inverse, we have $\gamma_0(y) > y$ for all $y \neq x_0$. Conjugating γ_0 by some element which does not fix x_0, we get an element γ_1 fixing some x_1 and such that $\gamma_1(y) > y$ for $y \neq x_1$. Assume for instance $x_0 < x_1$ and consider the element $\gamma = \gamma_0\gamma_1^{-1}$. Obviously, one has $\gamma(x_0) < x_0$ and $\gamma(x_1) > x_1$ and since we know that γ has at most one fixed point, γ must have a repulsive fixed point between x_0 and x_1 as we claimed.

Now, we can try to mimic the proof of Hölder's theorem. Consider two elements γ and γ' of Γ. Write $\gamma \preceq \gamma'$ if there is some $x \in \mathbf{R}$ such that $\gamma(y) \leq \gamma'(y)$ for all $y > x$. Clearly, our assumptions imply that this defines a total ordering on Γ which is left and right-invariant. Denote by Γ^+ the subset of elements of $\Gamma \setminus \{id\}$ such that $id \preceq \gamma$.

The next claim is a weak form of the archimedean property. *Fix some γ_0 in Γ^+ with a repulsive fixed point x_0, and let γ be any other element of Γ^+. Then there exists some positive integer k such that $\gamma \preceq \gamma_0^k$.* Indeed, choose some real numbers x_-, x_+ such that $x_- < x_0 < x_+$. For k big enough, one has $\gamma_0^k(x_-) < \gamma(x_-)$ and $\gamma_0^k(x_+) > \gamma(x_+)$ since x_0 is repulsive. It follows that $\gamma^{-1}\gamma_0^k$ has a fixed point in the interval $[x_-, x_+]$ which is therefore the unique fixed point of $\gamma^{-1}\gamma_0^k$. Hence we have $\gamma_0^k(y) > \gamma^{-1}(y)$ for all $y > x_+$ and $\gamma \preceq \gamma_0^k$. This proves our last claim.

Again, we fix some γ_0 in Γ^+ with a repulsive fixed point x_0. For each $\gamma \in \Gamma^+$ we define $\Phi(\gamma) \in \mathbf{N}$ to be the smallest integer k such that $\gamma \preceq \gamma_0^k$. If $\gamma^{-1} \in \Gamma^+$, we let $\Phi(\gamma) = -\Phi(\gamma^{-1})$ and finally we define $\Phi(id) = 0$. This defines a map $\Phi\colon \Gamma \to \mathbf{Z}$. Then we can copy from the proof of Hölder's theorem: Φ is a quasi-homomorphism and the limit $\phi(\gamma) = \lim_{n\to\infty} \Phi(\gamma^n)/n$ exists and defines a group homomorphism $\phi\colon \Gamma \to \mathbf{R}$.

It follows in particular that the first commutator group $[\Gamma, \Gamma]$ is contained in the kernel of ϕ. The final observation is that this kernel acts freely on the line. Otherwise, we saw that $Ker(\phi)$ would contain some element γ with a repulsive fixed point and we have already observed that this implies the existence of some integer k such that $\gamma_0 \preceq \gamma^k$ which in turn implies that $\phi(\gamma) \geq 1/k \neq 0$, a contradiction. Using Hölder's theorem, we conclude that $[\Gamma, \Gamma]$ is abelian.

We know the structure of free actions (of abelian groups) on the line: they are semi-conjugate to translation groups. More precisely, we know that there is a map $h\colon \mathbf{R} \to \mathbf{R}$ and an injective homomorphism $\psi\colon [\Gamma, \Gamma] \to \mathbf{R}$ which are such that for every $\gamma \in [\Gamma, \Gamma]$ and $x \in \mathbf{R}$, one has: $h(\gamma(x)) = h(x) + \psi(\gamma)$. If the image $\psi([\Gamma, \Gamma])$ is non discrete, this map h is unique up to post-composition by an affine map. So assume first that $\psi([\Gamma, \Gamma])$ is non discrete. Note that $[\Gamma, \Gamma]$ is a normal subgroup of Γ. It follows that for every γ in Γ, the map $h \circ \gamma$ coincides with h up to some affine map. This means precisely that h realizes a semi-conjugacy between Γ and some group of affine transformations of $\mathbf{R}$ and shows that Γ is indeed isomorphic to a subgroup of $\mathrm{Aff}(\mathbf{R})$. To finish the proof, we still have to show that $\psi([\Gamma, \Gamma])$ cannot be discrete, *i.e.* isomorphic to $\mathbf{Z}$. In this case, inner conjugacies by an element $\gamma \in \Gamma$ have to preserve the generator 1 of $\mathbf{Z}$ (the unique generator which is bigger that the identity in our ordering). This means that $\mathbf{Z}$ ($\simeq [\Gamma, \Gamma]$) lies in the center of Γ. This is not possible since for every fixed point x of an element γ of Γ, its orbit under $\mathbf{Z}$ would consist of fixed points of γ. $\square$

Hölder's theorem essentially characterizes translation groups as groups acting on the line with no fixed points. Solodov's theorem essentially characterizes groups of affine transformations as groups acting on the line with at most one fixed point. It is very tempting to try to prove a similar characterization of groups of projective transformations as groups acting on the circle with at most two fixed points... Unfortunately, this is not the case! N. Kovačević recently constructed a nice counter-example in [44].

THEOREM 6.13 (Kovačević). *There exists a finitely generated subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that every element different from the identity has at most two fixed points, such that all orbits are dense, and which is not conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbf{R})$.

Nevertheless, there is a very important characterization of groups which are conjugate to subgroups of $\mathrm{PSL}(2,\mathbf{R})$. This characterization is due to Casson-Jungreis and Gabai [15, 24], following earlier work of Tukia. We would have liked to include a discussion and a proof of this result, but that would be too long and we have to limit ourselves to a statement! Consider a sequence γ_n of elements of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Let us say that γ_n has the *convergence property* if it contains a subsequence γ_{n_k} which satisfies one of the following two properties:

- γ_{n_k} is equicontinuous;
- there exist two points x, y on the circle such that γ_{n_k} (resp. $\gamma_{n_k}^{-1}$) converges to a constant map on each compact interval in $\mathbf{S}^1 \setminus \{x\}$ (resp. in $\mathbf{S}^1 \setminus \{y\}$).

A subgroup Γ of $\mathrm{Homeo}_+(\mathbf{S}^1)$ is called a *convergence group* if every sequence of elements of Γ has the convergence property.

THEOREM 6.14 (Casson-Jungreis, Gabai). *A subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbf{R})$ *if and only if it is a convergence group.*

The reader should at least be able to prove the easy part of the theorem: subgroups of $\mathrm{PSL}(2,\mathbf{R})$ are convergence groups!

We revert now to groups acting on the circle. We state a general criterion which characterizes the bounded classes coming from some action.

THEOREM 6.15 ([25]). *Let* Γ *be a countable group and* c *a class in* $H_b^2(\Gamma, \mathbf{Z})$. *Then there exists a homomorphism* $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ *such that* $\phi^*(eu) = c$ *if and only if* c *can be represented by a cocycle which takes only the values* 0 *and* 1.

Proof. Of course, the necessary condition is clear from 6.3 and the main difficulty will be to construct some action from a cocycle taking two values. Let c be a 2-cocycle on the group Γ taking only the values 0 and 1. We saw that a central extension and a section lead to a 2-cocycle. The process can be reversed and we can construct a central extension $\widetilde{\Gamma}$ in the following way from a 2-cocycle c. As a set, $\widetilde{\Gamma}$ is the product $\mathbf{Z} \times \Gamma$ and we define a multiplication $\bullet$ by:

$$(n_1, \gamma_1) \bullet (n_2, \gamma_2) = (n_1 + n_2 + \bar{c}(\gamma_1, \gamma_2), \gamma_1\gamma_2)$$

where, as usual, $\bar{c}$ denotes the inhomogeneous cocycle associated to c. The fact that $\widetilde{\Gamma}$ is a group is a restatement of the fact that c is a cocycle. The projection $\widetilde{\Gamma} \to \Gamma$ is a group homomorphism.

Assume first that the cocycle c *is non degenerate*, *i.e.* that $\bar{c}(id, \gamma) = \bar{c}(\gamma, id) = 0$ for every γ in Γ (where id denotes the identity element in Γ). Then the identity element of $\widetilde{\Gamma}$ is $(0, id)$ and the map $n \in \mathbf{Z} \mapsto (n, id) \in \widetilde{\Gamma}$ is also a group homomorphism. Hence, we have a central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1\,.$$

The fact that c takes non negative values means that the subset P of $\widetilde{\Gamma}$ consisting of elements of the form (n, γ) with $n \geq 0$ is a semi-group, *i.e.* is stable under the product $\bullet$. Moreover, since c takes the values 0 and 1, the inverse of (n, γ) is $(-n, \gamma^{-1})$ or $(-n-1, \gamma^{-1})$. It follows that every element of $\widetilde{\Gamma}$ belongs to P or to its inverse. In other words, if one defines $\widetilde{\gamma}_1 \preceq \widetilde{\gamma}_2$ if $\widetilde{\gamma}_2\widetilde{\gamma}_1^{-1} \in P$ we get a total pre-order on $\widetilde{\Gamma}$ which is left invariant. Denote by $\mathbf{t}$ the element $(1, id)$ in $\widetilde{\Gamma}$. Note that for every $\widetilde{\gamma}$ in $\widetilde{\Gamma}$ we have $\widetilde{\gamma} \preceq \mathbf{t}\widetilde{\gamma}$.

The end of the proof mimics 6.8: One constructs a map $v\colon \widetilde{\Gamma} \to \mathbf{R}$ such that $\widetilde{\gamma}_1 \preceq \widetilde{\gamma}_2$ if and only if $v(\widetilde{\gamma}_1) \leq v(\widetilde{\gamma}_2)$ and such that $v(\widetilde{\gamma}\mathbf{t}) = v(\widetilde{\gamma}) + 1$ for every $\widetilde{\gamma} \in \widetilde{\Gamma}$. We may even choose v in such a way that the action of $\widetilde{\Gamma}$ on itself by left translations defines an action on $v(\widetilde{\Gamma}) \subset \mathbf{R}$ which extends to its closure. Then we extend this action of $\widetilde{\Gamma}$ to $\mathbf{R}$ using affine maps in the connected components of the complement of this closure. Finally, since $\mathbf{t}$ acts on $\mathbf{R}$ by the translation by 1, we get an action of the quotient group Γ on the circle $\mathbf{R}/\mathbf{Z}$. This construction was carried out in such a way that it is clear that the bounded Euler class of this action is precisely the class of the cocycle c.

Finally, we have to deal with the case of degenerate cocycles c. Note that the fact that c is a cocycle can be expressed by the identity:

$$\bar{c}(\gamma_1, \gamma_2) + \bar{c}(\gamma_1\gamma_2, \gamma_3) = \bar{c}(\gamma_2, \gamma_3) + \bar{c}(\gamma_1, \gamma_2\gamma_3).$$

It follows that there exists an integer $\nu = 0$ or 1 such that for every γ in Γ we have $\bar{c}(1, \gamma) = \bar{c}(\gamma, 1) = \nu$. The fact that c is degenerate means that $\nu = 1$. Then we can define $c' = 1 - c$. This is a new cocycle which is non degenerate and takes only the values 0 and 1. By the previous construction, we get an action of Γ on the circle corresponding to the bounded class of c'. Reversing the orientation of the circle, we get finally an action of Γ on the circle whose bounded Euler class is the class of c. □

6.6 Some examples

Recall that a group Γ is called *perfect* if every element is a product of commutators. It is *uniformly perfect* if there is an integer k such that every element is a product of at most k commutators. For such a uniformly perfect group, every quasi-homomorphism from Γ to $\mathbf{R}$ is bounded (since it is bounded on a single commutator) so that the canonical map from $H_b^2(\Gamma, \mathbf{R})$ to $H^2(\Gamma, \mathbf{R})$ is injective. Moreover the map from $H_b^2(\Gamma, \mathbf{Z})$ to $H_b^2(\Gamma, \mathbf{R})$ is also injective since there is no homomorphism from Γ to $\mathbf{R}$. In such a situation, the usual Euler class in $H^2(\Gamma, \mathbf{Z})$ determines the bounded Euler class, and therefore most of the topological dynamics of a group action.

An example of such a group is $\mathrm{SL}(n, \mathbf{Z})$ which is uniformly perfect for $n \geq 3$ and which, moreover is such that $H^2(\mathrm{SL}(n, \mathbf{Z}), \mathbf{Z}) = 0$ (for $n \geq 3$) [52]. As a corollary, we get immediately that *for $n \geq 3$, any action of* $\mathrm{SL}(n, \mathbf{Z})$ *on the circle has a fixed point.* This will be strengthened later in 7.1. Some other matrix groups have this property: see for instance [5, 14].

Consider the case of the Thompson group G. We can show that every element in G is a product of two commutators (see [28]) and that $H^2(G, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$. Using the Milnor-Wood inequality we can show that in $H^2(G, \mathbf{Z})$ only the elements $-1, 0, +1$ have a norm less than or equal to $1/2$. Hence we deduce that *any non-trivial action of the Thompson group G on the circle is semi-conjugate to the canonical action given by its embedding in* $\mathrm{PL}_+(\mathbf{S}^1)$ or to the reverse embedding obtained by conjugating by an orientation reversing homeomorphism of the circle (see [28] for more details).

Another situation where the bounded cohomology is easy to compute is the case of amenable groups. Let Γ be topological group (which will be frequently a discrete countable group) and denote by $C_b^0(\Gamma)$ the real vector space of bounded continuous functions on Γ with real values. We say that Γ is *amenable* if there is a linear operator $m\colon C_b^0(\Gamma) \to \mathbf{R}$ called a "mean" such that m is non negative on non negative elements, is equal to 1 on the constant function 1 and is invariant under left translations by elements of Γ. See the book [29] for a good description of the theory of these groups. Of course, compact groups are amenable: it suffices to define m as the integral over the Haar measure. Abelian groups are amenable. A closed subgroup of a locally compact amenable group is amenable and an increasing union of amenable groups is amenable. The category of amenable groups is also stable under extensions. In particular, solvable groups are amenable. The following is due to Johnson (see [39]).

THEOREM 6.16 (Johnson). *If Γ is an amenable group then its real bounded cohomology groups $H_b^k(\Gamma, \mathbf{R})$ are trivial for all $k \geq 0$.*

Proof. Strictly speaking, we only defined cohomology and bounded cohomology for discrete groups... but of course we could have done it for a general topological group. Since in any case we don't need this fact for non discrete groups, we assume Γ is a discrete amenable group equipped with a mean m. Let $c\colon \Gamma^{k+1} \to \mathbf{R}$ be a bounded k-cochain. Then we can define $\overline{m}(c)\colon \Gamma^k \to \mathbf{R}$ by taking the mean value with respect to the first variable. This linear operator $\overline{m}\colon C_b^k(\Gamma, \mathbf{R}) \to C_b^{k-1}(\Gamma, \mathbf{R})$ is an algebraic homotopy between the identity and 0, *i.e.* we have $d_{k-1}\overline{m} \pm \overline{m}d_k = id$. It implies immediately that a bounded cocycle is a bounded coboundary. $\square$

Let Γ be an amenable subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let $\widetilde{\Gamma}$ be the group of lifts in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$: this is also an amenable group since it is an extension of the amenable group $\mathbf{Z}$ by the amenable group Γ. The translation number map $\tau\colon \widetilde{\Gamma} \to \mathbf{R}$ is a quasi-homomorphism and is a homomorphism on one generator subgroups; the vanishing of bounded cohomology therefore implies that it is a homomorphism. *The rotation number is a homomorphism when restricted to an amenable group.*

If Γ is an amenable group, the group $H_b^2(\Gamma, \mathbf{Z})$ can easily be determined. Indeed, we know that $H_b^2(\Gamma, \mathbf{R}) = 0$ and that the kernel of the map from $H^2(\Gamma, \mathbf{Z})$ to $H^2(\Gamma, \mathbf{R})$ is the quotient group $H^1(\Gamma, \mathbf{R})/H^1(\Gamma, \mathbf{Z})$. We have therefore proved the following:

PROPOSITION 6.17. *Let Γ be an amenable group and $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ a homomorphism. Then the rotation number map $\rho \circ \phi\colon \Gamma \to \mathbf{R}/\mathbf{Z}$ is a homomorphism. If the image of this homomorphism is finite, then $\phi(\Gamma)$ has a finite orbit of the same cyclic structure. Otherwise, ϕ is semi-conjugate to the rotation group $\rho \circ \phi(\Gamma)$.*

Note that there is another approach to the proof of this proposition, using invariant probability measures. Indeed, let Γ be an amenable group acting on the circle by some homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$. If $u\colon \mathbf{S}^1 \to \mathbf{R}$ is a continuous function, we can consider the mean value of the bounded function $\gamma \in \Gamma \mapsto u(\phi(\gamma)(0))$. This gives a linear functional on the space of continuous functions u on the circle, equal to 1 on the function 1, *i.e.* this mean value has the form $\int_{\mathbf{S}^1} u\, d\mu$ for some probability measure μ on the circle. Of course this probability measure is invariant under $\phi(\Gamma)$. Assume now that μ has some non trivial atom, *i.e.* that some point x has some positive mass $\mu(\{x\}) > 0$. Then there is a finite number of atoms of the same mass so that we get a finite orbit for $\phi(\Gamma)$. If there is no atom, then there is a degree 1 map of the circle to itself which sends the measure μ to the Lebesgue measure since in this case the measure of an interval depends continuously on its endpoints. This map collapses each component of the complement of the support of μ to a point. This provides a semi-conjugacy of ϕ with a group of homeomorphisms preserving the Lebesgue measure, *i.e.* a rotation group. This gives another proof of Proposition 6.17.

Invariant probability measures also provide another definition of translation and rotation numbers. Let f be any element of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The qualitative description of the topological dynamics of f that we gave in 5.9 enables us to describe explicitly the probability measures μ on $\mathbf{S}^1$ which are invariant by f.

If the rotation number of f vanishes, the invariant probability measures are characterized by the fact that their support is contained in the fixed point set $Fix(f)$ of f. Indeed we know that the action of f on a connected component of the complement of $Fix(f)$ is conjugate to the translation by 1 on $\mathbf{R}$ and cannot preserve any non trivial finite measure.

If the rotation number is rational, invariant probability measures are concentrated on the set of periodic points.

If the rotation number is irrational and the orbits are dense, we know that f is conjugate to an irrational rotation. In this case, there is a unique invariant probability measure which is the image of the Lebesgue measure by the topological conjugacy (see [41]). If the orbits are not dense, there is an exceptional minimal set $K \subset \mathbf{S}^1$ and the support of any invariant probability has to coincide with K since we know that the connected components of $\mathbf{S}^1 - K$ are wandering intervals. In this case also there is a unique invariant probability μ which is the unique probability which maps to the Lebesgue measure by the degree 1 semi-conjugacy with a rotation.

Let $\widetilde{f}$ be an element of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ and μ a probability measure on $\mathbf{S}^1$ which is invariant by the corresponding homeomorphism of the circle $f = p(\widetilde{f})$. The function $\widetilde{f}(x) - x$ is $\mathbf{Z}$-periodic and therefore defines a function on $\mathbf{R}/\mathbf{Z}$ that we can integrate with respect to μ. It should be clear to the reader by now that the result is nothing more than the translation number $\tau(\widetilde{f})$. Suppose now that $\widetilde{f}$ and $\widetilde{g}$ are two elements of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $p(\widetilde{f})$ and $p(\widetilde{g})$ preserve the same measure μ. Note that $\widetilde{f}\widetilde{g}(x) - x = (\widetilde{f}(\widetilde{g}x) - \widetilde{g}(x)) + (\widetilde{g}(x) - x)$ and integrate with respect to μ. We get that $\tau(\widetilde{f}\widetilde{g}) = \tau(\widetilde{f}) + \tau(\widetilde{g})$. So we have proved the following:

PROPOSITION 6.18. *Let μ be a probability measure on the circle. Denote by* $\mathrm{Homeo}_+(\mathbf{S}^1, \mu)$ *the subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *consisting of homeomorphisms preserving μ. Then the rotation number* $\rho : \mathrm{Homeo}_+(\mathbf{S}^1, \mu) \to \mathbf{R}/\mathbf{Z}$ *is a homomorphism.*

Of course, in many situations the groups $H_b^2(\Gamma, \mathbf{R})$ can be infinite dimensional. For instance, this is the case of a free non abelian group, of the fundamental group of a closed orientable surface of genus $g > 1$ [4] and more generally of non elementary Gromov hyperbolic groups [19]. This is not a surprise since there are many homomorphisms from a free group for instance to $\mathrm{Homeo}_+(\mathbf{S}^1)$ and their bounded Euler classes are usually distinct.

In some cases, the bounded Euler class of a specific action on the circle might be useful to understand the structure of the group. Suppose for example that a group Γ is such that $H^1(\Gamma, \mathbf{R}) = H^2(\Gamma, \mathbf{R}) = 0$ and that we are given a homomorphism $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$. Then the image of the bounded Euler class $eu(\phi)$ in $H^2(\Gamma, \mathbf{Z})$ vanishes so that there is a (usually non bounded) quasi-homomorphism $\psi \colon \Gamma \to \mathbf{R}$ such that the bounded Euler cocycle $\phi^\star(c)$ is the coboundary of the 1-cochain $\psi(\gamma_1^{-1}\gamma_0)$. Modifying ψ by a bounded amount, we can assume that ψ is a homomorphism on one generator groups. With this condition, ψ is uniquely defined since we assumed that there is no homomorphism from Γ to $\mathbf{R}$. Of course, for any γ in Γ, the projection of $\psi(\gamma)$ in $\mathbf{R}/\mathbf{Z}$ is nothing more than the rotation number of $\phi(\gamma)$. Summing up, *with these algebraic conditions on the group* Γ, *any action of* Γ *on the circle determines canonically a quasi-homomorphism* $\psi\colon \Gamma \to \mathbf{R}$ *which is a lift of the rotation number map.*

A specific example is the modular group $\mathrm{PSL}(2,\mathbf{Z})$. As a group, it is isomorphic to the free product of two cyclic groups: $\mathrm{PSL}(2,\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/3\mathbf{Z}$ (see for instance [61]). Of course there is no non-trivial homomorphism from this group to $\mathbf{R}$ since it is generated by two elements of finite order. In the same way, its second real cohomology group is trivial (this follows for instance from the Mayer-Vietoris exact sequence since finite groups have trivial cohomology over the reals). We deduce that every action of $\mathrm{PSL}(2,\mathbf{Z})$ on the circle yields a well defined quasi-homomorphism $\psi\colon \mathrm{PSL}(2,\mathbf{Z}) \to \mathbf{R}$ lifting the rotation number. If we start with the canonical action of $\mathrm{PSL}(2,\mathbf{Z})$ on the circle $\mathbf{RP}^1$, the rotation numbers are not interesting: the only elliptic elements in $\mathrm{PSL}(2,\mathbf{Z})$ have order 2 and 3 so that the rotation number of elements in $\mathrm{PSL}(2,\mathbf{Z})$ are $0, 1/2, 1/3, 2/3 \in \mathbf{R}/\mathbf{Z}$. However the quasi-homomorphism $\Psi\colon \mathrm{PSL}(2,\mathbf{Z}) \to \mathbf{R}$ that we get is very interesting and has been studied in many different contexts: it is called the *Rademacher function.* The explicit formula giving Ψ as a function of the entries of a matrix in $\mathrm{PSL}(2,\mathbf{Z})$ involves the so called Dedekind sums which are important in number theory. We refer to [4] for a description of Ψ and a bibliography on this very nice subject.

7. Higher rank lattices

In this section, we study the problem of determining which lattices in semi-simple groups can act on the circle.

Let G be any Lie group and $\mathfrak{G}$ be its Lie algebra. The *real rank* of G is the maximal dimension of an abelian subalgebra $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ the linear operator $ad(a)\colon \mathfrak{G} \to \mathfrak{G}$ is diagonalizable over $\mathbf{R}$. For instance, the real rank of $\mathrm{SL}(n,\mathbf{R})$ is $n-1$: its Lie algebra consists of traceless matrices and contains the abelian diagonal traceless matrices. A *lattice* in a Lie group G is a discrete subgroup Γ such that the quotient G/Γ has finite measure with respect to a right invariant Haar measure. A lattice in a *semi-simple group* is called *reducible* if we can find two normal subgroups G_1, G_2 in G, connected and non trivial, which generate G, whose intersection is contained in the (discrete) center of G, and such that $(G_1 \cap \Gamma).(G_2 \cap \Gamma)$ has finite index