6.1 Group cohomology

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of Γ since, once again, Γ commutes with θ.

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of Γ on $\mathbf{S}^1$. As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in $[x, \theta(x)[$ is contractible, by definition. This means that any compact interval in $\mathbf{S}^{1'}$ is contractible and this implies (STRONG EXPANSIVITY).

We have now proved that if (MINIMALITY) *and* (EXPANSIVITY) *are both satisfied, then the group* Γ *must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group Γ. If a group Γ acts equicontinuously, then its closure in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a compact group by Ascoli's theorem. We analyzed compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$ in 4.1 : they turned out to be abelian and conjugate to groups of rotations.

We have shown that if (MINIMALITY) *holds then* Γ *is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let Γ be any subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let us use the structure theorem 5.6–5.8. If Γ is minimal, we have already proved the theorem. If Γ has a finite orbit, there is a Γ-invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the Γ-action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that Γ contains a non abelian free subgroup unless the restriction of the action of Γ to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a Γ-invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

6. BOUNDED EULER CLASS

6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let Γ be any group. Let us consider the (semi)-simplicial set $E\Gamma$ whose vertices are the elements of Γ and for which n-simplices are all $(n+1)$-tuples of elements of Γ. The i^{th} face of the simplex $(\gamma_0, \dots, \gamma_k)$ is $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$ where the term γ_i is omitted. Note that the set $E\Gamma$ does not depend on the group structure of Γ.

As a matter of fact, $E\Gamma$ is contractible since it is the full simplex over the set Γ. However, there is a simplicial free action of Γ on $E\Gamma$ induced by left translations of Γ on itself. Hence once could think of the quotient $B\Gamma$ of $E\Gamma$ by this action as a space whose fundamental group is Γ and with vanishing higher homotopy groups. One would like to define the cohomology of the group Γ as the cohomology of this quotient space $B\Gamma$. We should be careful with $B\Gamma$ since it has only one vertex (a group acts transitively on itself !).

However, guided by this idea, it is natural to define a *k-cochain* of Γ with values in some abelian group A as a map $c\colon \Gamma^{k+1} \to A$ which is *homogeneous*, *i.e.* such that $c(\gamma\gamma_0, \gamma\gamma_1, \dots, \gamma\gamma_k) = c(\gamma_0, \gamma_1, \dots, \gamma_k)$ identically. The set of these cochains is an abelian group denoted by $C^k(\Gamma, A)$. We have a natural coboundary d_k from $C^k(\Gamma, A)$ to $C^{k+1}(\Gamma, A)$ defined by

$$d_k c(\gamma_0, \dots, \gamma_{k+1}) = \sum_{i=0}^{k} (-1)^i c(\gamma_0, \dots, \widehat{\gamma_i}, \dots, \gamma_k)\,.$$

Of course, we have $d_{k+1} \circ d_k = 0$ and we define the *cohomology group* $H^k(\Gamma, A)$ as being the quotient of cocycles (*i.e.* the kernel of d_k) by coboundaries (*i.e.* the image of d_{k-1}). If A is moreover a ring, then there is a natural cup product from $H^k(\Gamma, A) \times H^l(\Gamma, A)$ to $H^{k+l}(\Gamma, A)$. We refer to [11] for an excellent account of this theory of group cohomology. Note that for any homomorphism ϕ from a group Γ to another group Γ', there is an induced homomorphism $\phi^\star\colon H^k(\Gamma', A) \to H^k(\Gamma, A)$.

A homogeneous map $c\colon \Gamma^{k+1} \to A$ can be written in a unique way in the form $c(\gamma_0, \dots, \gamma_k) = \overline{c}(\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, \dots, \gamma_{k-1}^{-1}\gamma_k)$ for a unique function $\overline{c}\colon \Gamma^k \to A$. Conversely, given a map $\overline{c}$ there is a unique homogeneous map c satisfying this relation. One says that $\overline{c}$ is the inhomogeneous cochain associated to c. In other words, the space $C^k(\Gamma, A)$ is canonically isomorphic to the A-module of all maps $\Gamma^k \to A$.

In degree 1, a cochain is a homogeneous map $c\colon \Gamma^2 \to A$ and the corresponding inhomogeneous cochain is a map $\overline{c}\colon \Gamma \to A$. It is interesting to check that c is a cocycle if and only if $\overline{c}$ is a homomorphism. Moreover 0-cochains are constant maps from Γ to A and their coboundary is therefore 0. It follows that *for any group* Γ, *the cohomology* $H^1(\Gamma, A)$ *is identified with the set of homomorphisms from* Γ *to* A.

In degree 2, the interpretation is quite interesting. Consider a central extension of Γ by A :

$$0 \longrightarrow A \xrightarrow{\ i\ } \widetilde{\Gamma} \xrightarrow{\ p\ } \Gamma \longrightarrow 1\,.$$

This means that $\widetilde{\Gamma}$ contains a subgroup isomorphic to A contained in its center and that the quotient by this subgroup is isomorphic to Γ. Suppose that the projection p has a section s which is a homomorphism from Γ to $\widetilde{\Gamma}$ such that $p \circ s = Id_\Gamma$. Then it follows that $\widetilde{\Gamma}$ is isomorphic to the direct product $\Gamma \times A$ by the homomorphism sending (γ, a) to $s(\gamma)i(a)$. Hence, in order to measure the non triviality of an extension we try to find the "obstruction" to finding a section s. This is done in the following way. Choose a set theoretical section s from Γ to $\widetilde{\Gamma}$; this is possible since p is onto. If γ_1 and γ_2 are two elements of Γ, consider $\bar{c}(\gamma_1, \gamma_2) = s(\gamma_1\gamma_2)^{-1}s(\gamma_1)s(\gamma_2)$. This element projects on the identity element of Γ under p since p is a homomorphism; it is therefore an element of the image of i and can be identified with an element of A. This defines a map $\bar{c}\colon \Gamma^2 \to A$. Let $c\colon \Gamma^3 \to A$ be the associated homogeneous cochain. One checks that c is a cocycle. Of course, the section s is not unique but another choice s' has the form $s'(\gamma) = s(\gamma)i(u(\gamma))$ for some function $u\colon \Gamma \to A$. If one computes the cocycle c' associated to this new choice of a section s', one finds that $c' - c$ is the coboundary of the 1-cochain associated to the map u. It follows that the cohomology class of c in $H^2(\Gamma, A)$ is well defined, *i.e.* does not depend on the choice of a section. This cohomology class is called the *Euler class of the extension* under consideration.

It is not difficult to check the following properties of the Euler class.

1) Two central extensions $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of A by Γ are isomorphic by some isomorphism which is the identity on the central subgroup A and inducing the identity on the quotient Γ if and only if they have the same Euler class in $H^2(\Gamma, A)$.

2) Any class in $H^2(\Gamma, A)$ corresponds to a central extension.

In short, $H^2(\Gamma, A)$ *parametrizes isomorphism classes of central extensions of* A *by* Γ.

Before coming back to the dynamics of groups acting on the circle, let us consider a few simple examples.

If $\Gamma = \mathbf{Z}$, it is clear that every extension admits a section which is a homomorphism: it suffices to choose arbitrarily $s(1)$ in $p^{-1}(1)$ and to define $s(n) = s(1)^n$ for $n \in \mathbf{Z}$. Hence, if $\Gamma = \mathbf{Z}$ or more generally if Γ is a free group, we have $H^2(\Gamma, A) = 0$.

Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$. It has a presentation of the form

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle .$$

Now consider the group $\widetilde{\Gamma}_g$ defined by the presentation

$$\widetilde{\Gamma}_g = \langle z, a_1, b_1, \dots, a_g, b_g \mid \\ a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \rangle .$$

The central subgroup A generated by z turns out to be infinite cyclic so that $\widetilde{\Gamma}_g$ defines a central extension of Γ_g by $\mathbf{Z}$, hence an Euler class in $H^2(\Gamma_g, \mathbf{Z})$. It is a fact that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic with $\mathbf{Z}$ and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus $g \geq 1$ has a contractible universal cover and that the cohomology of Γ_g can therefore be identified with the cohomology of the compact oriented surface of genus g (see [11] for more details).

6.2 THE EULER CLASS OF A GROUP ACTION ON THE CIRCLE

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1 .$$

The cohomology group $H^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ has been computed. It is isomorphic to $\mathbf{Z}$ and a generator is the Euler class of this central extension [50].

Consider now a homomorphism ϕ from some group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. Then, we can pull back the previous extension by ϕ. In other words, we consider the set of $(\gamma, \widetilde{f}) \in \Gamma \times \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi(\gamma) = p(\widetilde{f})$. This is a group $\widetilde{\Gamma}$ equipped with a canonical projection onto Γ whose kernel is isomorphic to $\mathbf{Z}$, *i.e.* $\widetilde{\Gamma}$ is a central extension of Γ by $\mathbf{Z}$. In case ϕ is injective, $\widetilde{\Gamma}$ is just the pre-image of $\phi(\Gamma)$ under p, which is the group of lifts of $\phi(\Gamma)$. The Euler class of this central extension of Γ is called *the Euler class of the homomorphism* ϕ and denoted by $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$. It is obviously a dynamical invariant in the sense that two conjugate homomorphisms ϕ_1 and ϕ_2 have the same Euler class in $H^2(\Gamma, \mathbf{Z})$. Note that it follows from the definition that $eu(\phi)$ is zero if and only if the homomorphism ϕ lifts to a homomorphism $\widetilde{\phi}\colon \Gamma \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi = p \circ \widetilde{\phi}$.

A few examples are in order. In the case of a single homeomorphism, *i.e.* when $\Gamma = \mathbf{Z}$, we saw that $H^2(\mathbf{Z}, \mathbf{Z}) = 0$. Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when Γ is free.

If Γ_g is the fundamental group of a closed oriented surface of genus $g \geq 1$, we know that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ so that the Euler class