6.6 SOME EXAMPLES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

Finally, we have to deal with the case of degenerate cocycles c. Note that the fact that c is a cocycle can be expressed by the identity:

$$\bar{c}(\gamma_1, \gamma_2) + \bar{c}(\gamma_1\gamma_2, \gamma_3) = \bar{c}(\gamma_2, \gamma_3) + \bar{c}(\gamma_1, \gamma_2\gamma_3).$$

It follows that there exists an integer $\nu = 0$ or 1 such that for every γ in Γ we have $\bar{c}(1, \gamma) = \bar{c}(\gamma, 1) = \nu$. The fact that c is degenerate means that $\nu = 1$. Then we can define $c' = 1 - c$. This is a new cocycle which is non degenerate and takes only the values 0 and 1. By the previous construction, we get an action of Γ on the circle corresponding to the bounded class of c'. Reversing the orientation of the circle, we get finally an action of Γ on the circle whose bounded Euler class is the class of c. □

6.6 Some examples

Recall that a group Γ is called *perfect* if every element is a product of commutators. It is *uniformly perfect* if there is an integer k such that every element is a product of at most k commutators. For such a uniformly perfect group, every quasi-homomorphism from Γ to $\mathbf{R}$ is bounded (since it is bounded on a single commutator) so that the canonical map from $H_b^2(\Gamma, \mathbf{R})$ to $H^2(\Gamma, \mathbf{R})$ is injective. Moreover the map from $H_b^2(\Gamma, \mathbf{Z})$ to $H_b^2(\Gamma, \mathbf{R})$ is also injective since there is no homomorphism from Γ to $\mathbf{R}$. In such a situation, the usual Euler class in $H^2(\Gamma, \mathbf{Z})$ determines the bounded Euler class, and therefore most of the topological dynamics of a group action.

An example of such a group is $\mathrm{SL}(n, \mathbf{Z})$ which is uniformly perfect for $n \geq 3$ and which, moreover is such that $H^2(\mathrm{SL}(n, \mathbf{Z}), \mathbf{Z}) = 0$ (for $n \geq 3$) [52]. As a corollary, we get immediately that *for $n \geq 3$, any action of* $\mathrm{SL}(n, \mathbf{Z})$ *on the circle has a fixed point.* This will be strengthened later in 7.1. Some other matrix groups have this property: see for instance [5, 14].

Consider the case of the Thompson group G. We can show that every element in G is a product of two commutators (see [28]) and that $H^2(G, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$. Using the Milnor-Wood inequality we can show that in $H^2(G, \mathbf{Z})$ only the elements $-1, 0, +1$ have a norm less than or equal to $1/2$. Hence we deduce that *any non-trivial action of the Thompson group G on the circle is semi-conjugate to the canonical action given by its embedding in* $\mathrm{PL}_+(\mathbf{S}^1)$ or to the reverse embedding obtained by conjugating by an orientation reversing homeomorphism of the circle (see [28] for more details).

Another situation where the bounded cohomology is easy to compute is the case of amenable groups. Let Γ be topological group (which will be frequently a discrete countable group) and denote by $C_b^0(\Gamma)$ the real vector space of bounded continuous functions on Γ with real values. We say that Γ is *amenable* if there is a linear operator $m\colon C_b^0(\Gamma) \to \mathbf{R}$ called a "mean" such that m is non negative on non negative elements, is equal to 1 on the constant function 1 and is invariant under left translations by elements of Γ. See the book [29] for a good description of the theory of these groups. Of course, compact groups are amenable: it suffices to define m as the integral over the Haar measure. Abelian groups are amenable. A closed subgroup of a locally compact amenable group is amenable and an increasing union of amenable groups is amenable. The category of amenable groups is also stable under extensions. In particular, solvable groups are amenable. The following is due to Johnson (see [39]).

THEOREM 6.16 (Johnson). *If Γ is an amenable group then its real bounded cohomology groups $H_b^k(\Gamma, \mathbf{R})$ are trivial for all $k \geq 0$.*

Proof. Strictly speaking, we only defined cohomology and bounded cohomology for discrete groups... but of course we could have done it for a general topological group. Since in any case we don't need this fact for non discrete groups, we assume Γ is a discrete amenable group equipped with a mean m. Let $c\colon \Gamma^{k+1} \to \mathbf{R}$ be a bounded k-cochain. Then we can define $\overline{m}(c)\colon \Gamma^k \to \mathbf{R}$ by taking the mean value with respect to the first variable. This linear operator $\overline{m}\colon C_b^k(\Gamma, \mathbf{R}) \to C_b^{k-1}(\Gamma, \mathbf{R})$ is an algebraic homotopy between the identity and 0, *i.e.* we have $d_{k-1}\overline{m} \pm \overline{m} d_k = id$. It implies immediately that a bounded cocycle is a bounded coboundary. □

Let Γ be an amenable subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let $\widetilde{\Gamma}$ be the group of lifts in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$: this is also an amenable group since it is an extension of the amenable group $\mathbf{Z}$ by the amenable group Γ. The translation number map $\tau\colon \widetilde{\Gamma} \to \mathbf{R}$ is a quasi-homomorphism and is a homomorphism on one generator subgroups; the vanishing of bounded cohomology therefore implies that it is a homomorphism. *The rotation number is a homomorphism when restricted to an amenable group.*

If Γ is an amenable group, the group $H_b^2(\Gamma, \mathbf{Z})$ can easily be determined. Indeed, we know that $H_b^2(\Gamma, \mathbf{R}) = 0$ and that the kernel of the map from $H^2(\Gamma, \mathbf{Z})$ to $H^2(\Gamma, \mathbf{R})$ is the quotient group $H^1(\Gamma, \mathbf{R})/H^1(\Gamma, \mathbf{Z})$. We have therefore proved the following:

PROPOSITION 6.17. *Let Γ be an amenable group and $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ a homomorphism. Then the rotation number map $\rho \circ \phi\colon \Gamma \to \mathbf{R}/\mathbf{Z}$ is a homomorphism. If the image of this homomorphism is finite, then $\phi(\Gamma)$ has a finite orbit of the same cyclic structure. Otherwise, ϕ is semi-conjugate to the rotation group $\rho \circ \phi(\Gamma)$.*

Note that there is another approach to the proof of this proposition, using invariant probability measures. Indeed, let Γ be an amenable group acting on the circle by some homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$. If $u\colon \mathbf{S}^1 \to \mathbf{R}$ is a continuous function, we can consider the mean value of the bounded function $\gamma \in \Gamma \mapsto u(\phi(\gamma)(0))$. This gives a linear functional on the space of continuous functions u on the circle, equal to 1 on the function 1, *i.e.* this mean value has the form $\int_{\mathbf{S}^1} u\,d\mu$ for some probability measure μ on the circle. Of course this probability measure is invariant under $\phi(\Gamma)$. Assume now that μ has some non trivial atom, *i.e.* that some point x has some positive mass $\mu(\{x\}) > 0$. Then there is a finite number of atoms of the same mass so that we get a finite orbit for $\phi(\Gamma)$. If there is no atom, then there is a degree 1 map of the circle to itself which sends the measure μ to the Lebesgue measure since in this case the measure of an interval depends continuously on its endpoints. This map collapses each component of the complement of the support of μ to a point. This provides a semi-conjugacy of ϕ with a group of homeomorphisms preserving the Lebesgue measure, *i.e.* a rotation group. This gives another proof of Proposition 6.17.

Invariant probability measures also provide another definition of translation and rotation numbers. Let f be any element of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The qualitative description of the topological dynamics of f that we gave in 5.9 enables us to describe explicitly the probability measures μ on $\mathbf{S}^1$ which are invariant by f.

If the rotation number of f vanishes, the invariant probability measures are characterized by the fact that their support is contained in the fixed point set *Fix*(f) of f. Indeed we know that the action of f on a connected component of the complement of *Fix*(f) is conjugate to the translation by 1 on $\mathbf{R}$ and cannot preserve any non trivial finite measure.

If the rotation number is rational, invariant probability measures are concentrated on the set of periodic points.

If the rotation number is irrational and the orbits are dense, we know that f is conjugate to an irrational rotation. In this case, there is a unique invariant probability measure which is the image of the Lebesgue measure by the topological conjugacy (see [41]). If the orbits are not dense, there is an exceptional minimal set $K \subset \mathbf{S}^1$ and the support of any invariant probability has to coincide with K since we know that the connected components of $\mathbf{S}^1 - K$ are wandering intervals. In this case also there is a unique invariant probability μ which is the unique probability which maps to the Lebesgue measure by the degree 1 semi-conjugacy with a rotation.

Let $\widetilde{f}$ be an element of $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ and μ a probability measure on $\mathbf{S}^1$ which is invariant by the corresponding homeomorphism of the circle $f = p(\widetilde{f})$. The function $\widetilde{f}(x) - x$ is $\mathbf{Z}$-periodic and therefore defines a function on $\mathbf{R}/\mathbf{Z}$ that we can integrate with respect to μ. It should be clear to the reader by now that the result is nothing more than the translation number $\tau(\widetilde{f})$. Suppose now that $\widetilde{f}$ and $\widetilde{g}$ are two elements of $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ such that $p(\widetilde{f})$ and $p(\widetilde{g})$ preserve the same measure μ. Note that $\widetilde{f}\widetilde{g}(x) - x = (\widetilde{f}(\widetilde{g}x) - \widetilde{g}(x)) + (\widetilde{g}(x) - x)$ and integrate with respect to μ. We get that $\tau(\widetilde{f}\widetilde{g}) = \tau(\widetilde{f}) + \tau(\widetilde{g})$. So we have proved the following:

PROPOSITION 6.18. *Let μ be a probability measure on the circle. Denote by* $\text{Homeo}_+(\mathbf{S}^1, \mu)$ *the subgroup of* $\text{Homeo}_+(\mathbf{S}^1)$ *consisting of homeomorphisms preserving μ. Then the rotation number* $\rho : \text{Homeo}_+(\mathbf{S}^1, \mu) \to \mathbf{R}/\mathbf{Z}$ *is a homomorphism.*

Of course, in many situations the groups $H_b^2(\Gamma, \mathbf{R})$ can be infinite dimensional. For instance, this is the case of a free non abelian group, of the fundamental group of a closed orientable surface of genus $g > 1$ [4] and more generally of non elementary Gromov hyperbolic groups [19]. This is not a surprise since there are many homomorphisms from a free group for instance to $\text{Homeo}_+(\mathbf{S}^1)$ and their bounded Euler classes are usually distinct.

In some cases, the bounded Euler class of a specific action on the circle might be useful to understand the structure of the group. Suppose for example that a group Γ is such that $H^1(\Gamma, \mathbf{R}) = H^2(\Gamma, \mathbf{R}) = 0$ and that we are given a homomorphism $\phi\colon \Gamma \to \text{Homeo}_+(\mathbf{S}^1)$. Then the image of the bounded Euler class $eu(\phi)$ in $H^2(\Gamma, \mathbf{Z})$ vanishes so that there is a (usually non bounded) quasi-homomorphism $\psi\colon \Gamma \to \mathbf{R}$ such that the bounded Euler cocycle $\phi^\star(c)$ is the coboundary of the 1-cochain $\psi(\gamma_1^{-1}\gamma_0)$. Modifying ψ by a bounded amount, we can assume that ψ is a homomorphism on one generator groups. With this condition, ψ is uniquely defined since we assumed that there is no homomorphism from Γ to $\mathbf{R}$. Of course, for any γ in Γ, the projection of $\psi(\gamma)$ in $\mathbf{R}/\mathbf{Z}$ is nothing more than the rotation number of $\phi(\gamma)$. Summing up, *with these algebraic conditions on the group* Γ, *any action of* Γ *on the circle determines canonically a quasi-homomorphism* $\psi \colon \Gamma \to \mathbf{R}$ *which is a lift of the rotation number map.*

A specific example is the modular group $\mathrm{PSL}(2, \mathbf{Z})$. As a group, it is isomorphic to the free product of two cyclic groups: $\mathrm{PSL}(2, \mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/3\mathbf{Z}$ (see for instance [61]). Of course there is no non-trivial homomorphism from this group to $\mathbf{R}$ since it is generated by two elements of finite order. In the same way, its second real cohomology group is trivial (this follows for instance from the Mayer-Vietoris exact sequence since finite groups have trivial cohomology over the reals). We deduce that every action of $\mathrm{PSL}(2, \mathbf{Z})$ on the circle yields a well defined quasi-homomorphism $\psi \colon \mathrm{PSL}(2, \mathbf{Z}) \to \mathbf{R}$ lifting the rotation number. If we start with the canonical action of $\mathrm{PSL}(2, \mathbf{Z})$ on the circle $\mathbf{RP}^1$, the rotation numbers are not interesting: the only elliptic elements in $\mathrm{PSL}(2, \mathbf{Z})$ have order 2 and 3 so that the rotation number of elements in $\mathrm{PSL}(2, \mathbf{Z})$ are $0, 1/2, 1/3, 2/3 \in \mathbf{R}/\mathbf{Z}$. However the quasi-homomorphism $\Psi \colon \mathrm{PSL}(2, \mathbf{Z}) \to \mathbf{R}$ that we get is very interesting and has been studied in many different contexts: it is called the *Rademacher function*. The explicit formula giving Ψ as a function of the entries of a matrix in $\mathrm{PSL}(2, \mathbf{Z})$ involves the so called Dedekind sums which are important in number theory. We refer to [4] for a description of Ψ and a bibliography on this very nice subject.

7. Higher rank lattices

In this section, we study the problem of determining which lattices in semi-simple groups can act on the circle.

Let G be any Lie group and $\mathfrak{G}$ be its Lie algebra. The *real rank* of G is the maximal dimension of an abelian subalgebra $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ the linear operator $ad(a) \colon \mathfrak{G} \to \mathfrak{G}$ is diagonalizable over $\mathbf{R}$. For instance, the real rank of $\mathrm{SL}(n, \mathbf{R})$ is $n-1$: its Lie algebra consists of traceless matrices and contains the abelian diagonal traceless matrices. A *lattice* in a Lie group G is a discrete subgroup Γ such that the quotient G/Γ has finite measure with respect to a right invariant Haar measure. A lattice in a *semi-simple group* is called *reducible* if we can find two normal subgroups G_1, G_2 in G, connected and non trivial, which generate G, whose intersection is contained in the (discrete) center of G, and such that $(G_1 \cap \Gamma).(G_2 \cap \Gamma)$ has finite index