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LATTICES OF COVARIANT QUADRATIC FORMS

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1. INTRODUCTION

The problem of constructing integral lattices in Euclidean space with big density for the associated sphere packing has attracted considerable attention in the last years; cf. [CoS88]. Some of the lattices found in this context were constructed as G-lattices for some finite group G; cf. [NeP95], [Neb95], [Neb96a], [Neb96b], or [Ple98] for a survey. Other sources of constructions were lattices associated with number fields or semi-simple algebras; cf. [BaM94]. Rather than looking at just one bilinear form on a lattice, the present investigation is geared towards the study of certain families of such forms.

More precisely, a rather general and flexible setting for the $\mathbf{Z}$-lattice $\mathrm{Bil}_{\mathbf{Z}G}(L)$ of all integral G-invariant bilinear forms on a $\mathbf{Z}G$-lattice L is given: one replaces the group ring $\mathbf{Z}G$ by a $\mathbf{Z}$-order Λ with a positive involution and the invariant bilinear forms by covariant ones, as defined in Chapter 2. One learns from [Opg96] and [Opg01] that one should look at the dual lattice L^* at the same time. As pointed out by J.-P. Tignol, the endomorphism ring $\mathrm{End}_\Lambda(L \oplus L^*)$ accommodates both the integral bilinear covariant forms on L and on its dual L^*. Even if the two orders Λ_i with involutions and lattices L_i are completely different, it now becomes natural to consider that the two lattices $\mathrm{Bil}_{\Lambda_1}(L_1)$ and $\mathrm{Bil}_{\Lambda_2}(L_2)$ of bilinear covariant forms on Λ_i-lattices L_i are equivalent if the endomorphism rings $\mathrm{End}_{\Lambda_i}(L_i \oplus L_i^*)$ are isomorphic; cf. Chapter 2 for a more precise definition.

In this way the lattice of all integral bilinear forms on the $\mathbf{Z}$-lattice $\mathbf{Z}^n$ becomes equivalent to $\mathrm{Bil}_{\mathbf{Z}G}(\bigoplus^n M)$ for any absolutely irreducible $\mathbf{Z}G$-lattice M admitting a unimodular G-invariant bilinear form. However, the situation is more interesting if the endomorphism rings $\mathrm{End}_\Lambda(L \oplus L^*)$ are not maximal orders, for instance hereditary, to mention the next simplest case. In Chapter 4 a canonical process is described which associates with each $\mathrm{Bil}_\Lambda(L)$ a Λ-lattice $\bar{L}$ such that $\mathrm{End}_\Lambda(\bar{L} \oplus \bar{L}^*)$ is hereditary. At the same time one gets an invariant called the e-$*$-depth measuring how far away L, resp. $\mathrm{Bil}_\Lambda(L)$, is from this well behaved situation.

This process generalizes Watson's process for constructing elementary quadratic forms out of arbitrary integral quadratic forms; cf. [Wat62] (where 'elementary' means that the exponent of the discriminant group is square free). Indeed, the present investigation can also be viewed as a generalization of the study of a single positive definite integral bilinear form ϕ, at least if ϕ is primitive, i. e. surjective onto $\mathbf{Z}$, namely by obtaining $\mathbf{Z}\phi$ as $\mathrm{Bil}_\Lambda(L)$. Equivalence then means that the exponents of the discriminant groups (= biggest elementary divisors of the Gram matrices) are equal for the two primitive forms considered. It should be noted that the general procedure applied here is called the *radical idealizer process* and is quite common in the general theory of orders.

In Chapter 3 the group of autoequivalences is studied without using the underlying lattice L in any serious way other than via $\mathrm{End}_\Lambda(L \oplus L^*)$. The notions depth and $*$-depth for $\mathrm{Bil}_\Lambda(L)$ measure how far $\mathrm{End}_\Lambda(L)$ and $\mathrm{End}_\Lambda(L \oplus L^*)$ are away from being hereditary. The $*$-depth zero situations are often classifiable. In the depth zero situation structural results on the outer group of autoequivalences can be given. Even more restrictions for the outer automorphism group in the $*$-depth zero case are given in Chapter 4.

Chapter 5 studies the special situation where inversion of nondegenerate forms can be rescaled to become a $\mathbf{Z}$-linear mapping of the nondegenerate elements in $\mathrm{Bil}_\Lambda(L)$ into $\mathrm{Bil}_\Lambda(L^*)$. Quebbemann's definition of modular lattices, cf. [Que95] and [Que97], is taken up to define $\mathrm{Bil}_\Lambda(L)$ to be modular if there is a simultaneous modularity transformation for all positive definite forms in $\mathrm{Bil}_\Lambda(L)$. Finally, in Chapter 6, some examples are studied, e. g. if $\mathrm{End}_\Lambda(L)$ is a $\mathbf{Z}$-order in the algebra $\mathbf{Q}^{2\times 2}$. Examples of this nature have also been studied by Bavard, cf. [Bav97], in a geometric manner in the context of symplectic lattices.

Whenever something new is introduced, one should justify it by giving the benefits for the old problems. So, for instance, the present investigations give a better understanding of the normalizer of a finite unimodular group within the full unimodular group (cf. discussion of $N(L)$ following Definition 4.4). The sort of insight one gets into the structure of the normalizer allows one to compare normalizers in their actions on the $\mathrm{Bil}_\Lambda(L)$ even if the groups are of different degrees.

In this sense the examples at the end of the paper describe infinitely many normalizers. The reader who wants to look at some other, more concrete, examples might use the package [1]) *CARAT*® handling low-dimensional crystallographic groups; cf. [OPS98] or [PS00]. Here are some further applications of the present investigations: they help to check when two finite unimodular groups are conjugate in the full unimodular group by comparing the lattices of invariant forms; they help to create models of such lattices in low dimensions by passing to equivalent lattices of covariant forms; they help to find candidates for lattices of covariant forms which contain interesting positive definite bilinear forms, and to locate these forms inside the lattice of covariant forms.

It is a pleasure to acknowledge many inspiring discussions with G. Nebe.

2. Covariant forms and equivalence

Throughout the paper, $\mathcal{A}$ denotes a semi-simple $\mathbf{Q}$-algebra with a positive involution $^\circ$, i.e. an antiautomorphism of order two of $\mathcal{A}$ such that $\mathcal{A} \to \mathbf{Q}$: $a \mapsto tr_{\mathcal{A}/\mathbf{Q}}(aa^\circ)$ is a positive definite quadratic form on $\mathcal{A}$, where $tr_{\mathcal{A}/\mathbf{Q}}$ denotes the reduced trace of $\mathcal{A}$. Together with $\mathcal{A}$, fix a faithful finite dimensional right $\mathcal{A}$-module $\mathcal{V}$. The basic data to start with are $\mathcal{A}$, $^\circ$, and L, where L is a full $\mathbf{Z}$-lattice in $\mathcal{V} = L_\mathbf{Q} := \mathbf{Q} \otimes_\mathbf{Z} L$. Because of the involution, $\mathcal{V}^* := \mathrm{Hom}_\mathbf{Q}(\mathcal{V}, \mathbf{Q})$ becomes a right $\mathcal{A}$-module again, which is isomorphic to $\mathcal{V}$. Inside $\mathcal{V}^*$ one has $L^* := \{\varphi \in \mathcal{V}^* \mid L\varphi \subset \mathbf{Z}\}$, which can be identified with $\mathrm{Hom}_\mathbf{Z}(L, \mathbf{Z})$.

Definition 2.1.

(i) $\Lambda(L) := \{a \in \mathcal{A} \mid La \subseteq L \text{ and } L^*a \subseteq L^*\}$ is called the $^\circ$-*invariant order* of L in $\mathcal{A}$.

(ii) A $\mathbf{Z}$-bilinear form $\phi\colon L \times L \to \mathbf{Z}$ is called *covariant* (with respect to $^\circ$) if it satisfies

$$\phi(Va, W) = \phi(V, Wa^\circ) \text{ for all } V, W \in L,\ a \in \Lambda\,,$$

where Λ is any $^\circ$-invariant $\mathbf{Z}$-order in $\mathcal{A}$, contained in $\Lambda(L)$ of finite index.

(iii) The $\mathbf{Z}$-lattice of all, resp. all symmetric or skew-symmetric, covariant $\mathbf{Z}$-bilinear forms on L is denoted by $\mathrm{Bil}_\Lambda(L)$, resp. $\mathrm{Bil}^+_\Lambda(L)$ or $\mathrm{Bil}^-_\Lambda(L)$. Finally $\mathrm{Bil}^+_{\Lambda,>0}(L)$ denotes the set of positive definite elements in $\mathrm{Bil}^+_\Lambda(L)$.

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[1]) This is available via internet http://wwwb.math.rwth-aachen.de/carat/index.html.