Zeitschrift: L'Enseignement Mathématique

Band: 47 (2001)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: GROUPS ACTING ON THE CIRCLE

Kapitel: 7.1 WITTE'S THEOREM

Autor: GHYS, Étienne

DOI: https://doi.org/10.5169/seals-65441

in Γ. Otherwise, we say that Γ is *irreducible*. Note that lattices in simple Lie groups are obviously irreducible.

The first example of a lattice is $\mathrm{SL}(n,\mathbf{Z})$ in $\mathrm{SL}(n,\mathbf{R})$: the corresponding quotient has finite volume (but is not compact).

Another example to keep in mind is the following. Consider the field $\mathbf{Q}(\sqrt{2})$ and its ring of integers $\mathcal{O}=\mathbf{Z}[\sqrt{2}]$. The field $\mathbf{Q}(\sqrt{2})$ has two embeddings in $\mathbf{R}$ given by $a+b\sqrt{2}\in\mathbf{Q}(\sqrt{2})\mapsto a\pm b\sqrt{2}\in\mathbf{R}$. This gives two embeddings of the group $\mathrm{SL}(2,\mathcal{O})$ in $\mathrm{SL}(2,\mathbf{R})$. The images of these embeddings are dense but the embedding of $\mathrm{SL}(2,\mathcal{O})$ in $\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R})$ has a discrete image which is an irreducible lattice in $\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R})$ (whose real rank is 2). Of course, we can construct many more examples using this kind of arithmetic construction: Borel showed for instance that any semi-simple Lie group (with no compact factor) contains at least an irreducible lattice (and even a cocompact one).

Note also that if a compact oriented manifold M of dimension n admits a metric with constant negative curvature, its universal cover is identified with the hyperbolic space H^n of dimension n. It follows that the fundamental group Γ of M is a discrete cocompact subgroup of the group of positive isometries of H^n which is the simple Lie group $\mathrm{SO}_0(n,1)$. These examples provide lattices in real rank 1 simple Lie groups.

For the theory of lattices in Lie groups, we refer to [48, 72].

7.1 WITTE'S THEOREM

In [70], Witte proves the following remarkable theorem:

THEOREM 7.1 (Witte). *Let Γ be a finite index subgroup of* $\mathrm{SL}(n,\mathbf{Z})$ *for $n\geq 3$. Then any homomorphism $\phi\colon\Gamma\to$* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has a finite image.*

The proof will be derived from the following

THEOREM 7.2 (Witte). *A finite index subgroup of* $\mathrm{SL}(n,\mathbf{Z})$ *for $n\geq 3$ is not left orderable.*

Proof. It suffices to prove it for a finite index subgroup Γ of $\mathrm{SL}(3,\mathbf{Z})$ since a subgroup of a left ordered group is of course left ordered. Suppose by contradiction that there is a left invariant total order $\preceq$ on Γ. Choose some integer $k\geq 1$ so that the following six elementary matrices belong to Γ:

$$a_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \quad a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}.$$

It is easy to check the following relations between these matrices. Taking indices modulo 6, for every i the matrices a_i and a_{i+1} commute and the commutator of a_{i-1} and a_{i+1} is $a_i^{\pm k}$. Fix some i and let us analyze the structure of $\preceq$ on the group H_i generated by a_{i-1}, a_i, a_{i+1}. Allowing ourself to replace a_{i-1} or a_{i+1} by their inverses and to permute them, we can define three elements α, β, γ such that $\{\alpha, \beta\} = \{a_{i-1}^{\pm 1}, a_{i+1}^{\pm 1}\}$ and $\gamma = a_i{}^{\pm k}$ and such that the following conditions are satisfied:

$$\alpha\gamma = \gamma\alpha \quad ; \quad \beta\gamma = \gamma\beta \quad ; \quad \alpha\beta\alpha^{-1}\beta^{-1} = \gamma^{-1}$$

$$1 \prec \alpha \quad ; \quad 1 \prec \beta \quad ; \quad 1 \prec \gamma$$

(1 denotes the identity element). If ξ is an element of Γ, we set $|\xi| = \xi$ if $1 \preceq \xi$ and ξ^{-1} otherwise. If two elements ξ, ζ in Γ are such that $1 \prec \xi$ and $1 \prec \zeta$, we write $\xi \ll \zeta$ if for every integer $n \geq 1$, we have $\xi^n \prec \zeta$. We claim that $\gamma \ll \alpha$ or $\gamma \ll \beta$ (which implies that $|a_i| \ll |a_{i-1}|$ or $|a_i| \ll |a_{i+1}|$). Indeed, suppose that there is some integer $n \geq 1$ such that $\alpha \prec \gamma^n$ and $\beta \prec \gamma^n$ and let us compute

$$\delta_m = \alpha^m \beta^m (\alpha^{-1}\gamma^n)^m (\beta^{-1}\gamma^n)^m .$$

Since δ_m is a product of elements in Γ which are bigger than 1, we have $1 \prec \delta_m$. Now the product defining δ_m can easily be estimated since we know that γ commutes with α and β and that interchanging the order of an α and a β is compensated by the introduction of a γ. We find

$$\delta_m = \gamma^{-m^2+2mn} .$$

Since $1 \prec \gamma$, we know that γ to a negative power is less than 1. For m big enough, we get $\delta_m \prec 1$. This is a contradiction.

Coming back to our six matrices a_i, we find that $|a_i| \ll |a_{i-1}|$ or $|a_i| \ll |a_{i+1}|$. If we assume for instance $|a_1| \ll |a_2|$, we therefore deduce cyclically $|a_1| \ll |a_2| \ll |a_3| \ll |a_4| \ll |a_5| \ll |a_6| \ll |a_1|$, and this is a contradiction. □

Let us now prove Theorem 7.1 using similar ideas. Of course, Theorem 7.2 means that a finite index subgroup of $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 3$ does not act faithfully on the line (by orientation preserving homeomorphisms).

Consider first a torsion free finite index subgroup Γ of $SL(3, \mathbf{Z})$ and suppose by contradiction that there is an action $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ with infinite image. According to an important theorem, due to Margulis, every normal subgroup of a lattice in a simple Lie group of rank at least 2 is either of finite index or is finite (see [48, 64]). It follows that the action ϕ is faithful.

As in the proof of Theorem 7.2, choose an integer k such that the matrices $(a_i)_{i=1\dots 6}$ are in Γ. Note that the group H_i generated by a_{i-1}, a_i, a_{i+1} is nilpotent, hence amenable, so that the rotation number is a homomorphism when restricted to H_i. Since $a_i^{\pm k}$ is a commutator, it follows that the rotation numbers of all $\phi(a_i)$ vanish. Define A_i as being the unique lift of $\phi(a_i)$ whose translation number is 0. We claim that the elements A_i of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ also satisfy the relations that for every i the homeomorphisms A_i and A_{i+1} commute and the commutator of A_{i-1} and A_{i+1} is $A_i^{\pm k}$. Indeed $A_i A_{i+1} A_i^{-1} A_{i+1}^{-1}$ and $A_{i+1} A_{i-1} A_{i+1}^{-1} A_{i-1}^{-1} A_i^{\mp k}$ project on the identity and have translation number 0 since the inverse image of H_i in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is nilpotent and the restriction of τ to this group is a homomorphism. Consider now the (left ordered) group of homeomorphisms of the line generated by the A_i. We can reproduce exactly the same argument that we used in Theorem 7.2 to get a contradiction.

Consider finally the general case of an action $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ of a finite index subgroup of $SL(n, \mathbf{Z})$ $(n \geq 3)$. Replacing Γ by a finite index subgroup, we can assume that Γ is torsion free. Of course, $SL(3, \mathbf{Z})$ is the subgroup of $SL(n, \mathbf{Z})$ consisting of matrices preserving $\mathbf{Z}^3 \simeq \mathbf{Z}^3 \times \{0\} \subset \mathbf{Z}^n$ and Γ intersects $SL(3, \mathbf{Z})$ on a subgroup of finite index in $SL(3, \mathbf{Z})$. Since we have already dealt with the case $n = 3$, the kernel of ϕ contains a subgroup of finite index in the infinite group $\Gamma \cap SL(3, \mathbf{Z})$. By the theorem of Margulis that we mentioned, the kernel of ϕ is a subgroup of finite index in Γ so that the image of ϕ is a finite group. Theorem 7.1 is proved.

It turns out that the arguments used in this proof can be extended to a family of lattices more general than finite index subgroups of $SL(n, \mathbf{Z})$ for $n \geq 3$. The general situation in which Witte proves his theorem is for *arithmetic* lattices in algebraic semi-simple groups of $\mathbf{Q}$-rank at least 2. We will not define this concept and refer to the original article by Witte. Note however that the method of proof cannot be generalized to an arbitrary lattice since it uses strongly the existence of nilpotent subgroups (which don't exist for example if the lattice is cocompact). However, this strongly suggests the following:

PROBLEM 7.3. *Is it true that no lattice in a simple Lie group of real rank at least 2 is left orderable?*