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of different degrees.

In this sense the examples at the end of the paper describe infinitely many normalizers. The reader who wants to look at some other, more concrete, examples might use the package[1]) *CARAT*® handling low-dimensional crystallographic groups; cf. [OPS98] or [PS00]. Here are some further applications of the present investigations: they help to check when two finite unimodular groups are conjugate in the full unimodular group by comparing the lattices of invariant forms; they help to create models of such lattices in low dimensions by passing to equivalent lattices of covariant forms; they help to find candidates for lattices of covariant forms which contain interesting positive definite bilinear forms, and to locate these forms inside the lattice of covariant forms.

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2. Covariant forms and equivalence

Throughout the paper, $\mathcal{A}$ denotes a semi-simple $\mathbf{Q}$-algebra with a positive involution $^\circ$, i.e. an antiautomorphism of order two of $\mathcal{A}$ such that $\mathcal{A} \to \mathbf{Q}$: $a \mapsto tr_{\mathcal{A}/\mathbf{Q}}(aa^\circ)$ is a positive definite quadratic form on $\mathcal{A}$, where $tr_{\mathcal{A}/\mathbf{Q}}$ denotes the reduced trace of $\mathcal{A}$. Together with $\mathcal{A}$, fix a faithful finite dimensional right $\mathcal{A}$-module $\mathcal{V}$. The basic data to start with are $\mathcal{A}$, $^\circ$, and L, where L is a full $\mathbf{Z}$-lattice in $\mathcal{V} = L_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} L$. Because of the involution, $\mathcal{V}^* := \mathrm{Hom}_{\mathbf{Q}}(\mathcal{V}, \mathbf{Q})$ becomes a right $\mathcal{A}$-module again, which is isomorphic to $\mathcal{V}$. Inside $\mathcal{V}^*$ one has $L^* := \{\varphi \in \mathcal{V}^* \mid L\varphi \subset \mathbf{Z}\}$, which can be identified with $\mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$.

Definition 2.1.

(i) $\Lambda(L) := \{a \in \mathcal{A} \mid La \subseteq L \text{ and } L^*a \subseteq L^*\}$ is called the $^\circ$*-invariant* order of L in $\mathcal{A}$.

(ii) A $\mathbf{Z}$-bilinear form $\phi\colon L \times L \to \mathbf{Z}$ is called *covariant* (with respect to $^\circ$) if it satisfies

$$\phi(Va, W) = \phi(V, Wa^\circ) \text{ for all } V, W \in L,\ a \in \Lambda\,,$$

where Λ is any $^\circ$-invariant $\mathbf{Z}$-order in $\mathcal{A}$, contained in $\Lambda(L)$ of finite index.

(iii) The $\mathbf{Z}$-lattice of all, resp. all symmetric or skew-symmetric, covariant $\mathbf{Z}$-bilinear forms on L is denoted by $\mathrm{Bil}_\Lambda(L)$, resp. $\mathrm{Bil}_\Lambda^+(L)$ or $\mathrm{Bil}_\Lambda^-(L)$. Finally $\mathrm{Bil}_{\Lambda,>0}^+(L)$ denotes the set of positive definite elements in $\mathrm{Bil}_\Lambda^+(L)$.

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[1]) This is available via internet http://wwwb.math.rwth-aachen.de/carat/index.html.

Extending this notation for any commutative ring R containing $\mathbf{Z}$, one can consider covariant R-valued bilinear forms. They give rise to the R-modules $\mathrm{Bil}_{\Lambda_R}(L_R)$, resp. $\mathrm{Bil}^+_{\Lambda_R}(L_R)$ and $\mathrm{Bil}^-_{\Lambda_R}(L_R)$, spanned by the above $\mathbf{Z}$-lattices. If R is contained in $\mathbf{R}$, $\mathrm{Bil}^+_{\Lambda_R,>0}(L_R)$ denotes the set of positive definite elements in $\mathrm{Bil}^+_{\Lambda_R}(L_R)$. One checks, that $\mathrm{Bil}^+_{\Lambda_\mathbf{R},>0}(L_\mathbf{R})$ is an open, nonempty cone in the real vector space $\mathrm{Bil}^+_{\Lambda_\mathbf{R}}(L_\mathbf{R})$. Any nondegenerate element of $\mathrm{Bil}_{\mathcal{A}}(L_\mathbf{Q})$ can be used to recover the involution $^\circ$ on $\mathcal{A}$. To connect covariance with the more familiar notion of a sesquilinear form – cf. [Scha85], p. 236, [BaF96] –, one should note that composition with the reduced trace of $\mathcal{A}$ yields a $\mathbf{Z}$-isomorphism of the lattice of sesquilinear maps of L taking values in the inverse different of $\Lambda(L)$ onto $\mathrm{Bil}_\Lambda(L)$. Three typical examples will demonstrate the generality of the concept:

EXAMPLE 2.2.

(i) Fix a positive definite symmetric matrix $f \in \mathbf{Q}^{n\times n}$. Let $\mathcal{A} = \mathbf{Q}^{n\times n}$ with $a^\circ = fa^{tr}f^{-1}$ for all $a \in \mathcal{A}$ and let $L = \mathbf{Z}^{1\times n}$. There is a unique positive definite rational multiple f_0 of f, which is *integral* and *primitive*, i. e. the greatest common divisor of the entries of f_0 is 1. One checks that $\mathrm{Bil}_\Lambda(L) = \mathbf{Z}f_0$ and $\mathrm{Bil}^+_{\Lambda,>0}(L) = \mathbf{N}f_0$. If f_0 is unimodular, then $\Lambda(L) = \mathbf{Z}^{n\times n}$, any other $\Lambda(L)$-lattice is of the form $\bigoplus^k L$, and $\mathrm{Bil}_\Lambda(\bigoplus^k L) = \{x \otimes f_0 \mid x \in \mathbf{Z}^{k\times k}\}$, where $\otimes$ denotes the Kronecker product (of two matrices). Note that $\mathrm{End}_{\Lambda(L)}(\bigoplus^k L) \cong \mathbf{Z}^{k\times k}$.

(ii) Let $G \leq \mathrm{GL}_n(\mathbf{Z})$ be a finite unimodular group. Set $\mathcal{A} := \overline{\mathbf{Q}G}$ the enveloping algebra of G, i. e. the subalgebra of $\mathbf{Q}^{n\times n}$ spanned by the matrices of G (clearly an epimorphic image of the group algebra $\mathbf{Q}G$) and let $L := \mathbf{Z}^{1\times n}$. Obviously the standard involution $g \mapsto g^{-1}$ for $g \in G$ of $\mathbf{Q}G$ induces a positive involution on $\overline{\mathbf{Q}G}$. The order $\Lambda(L)$ contains $\overline{\mathbf{Z}G}$, the $\mathbf{Z}$-span of the matrices of G as a suborder of finite index. $\mathrm{Bil}_\Lambda(L)$ consists of all G-invariant bilinear forms. $\mathrm{Bil}^+_{\Lambda_\mathbf{R},>0}(L_\mathbf{R})$ is known as the *Bravais manifold* of G. If there is no finite unimodular group H containing G properly with the same $\mathrm{Bil}_\Lambda(L)$, resp. $\mathrm{Bil}^+_\Lambda(L)$, then G is called the *strict Bravais group*, resp. the *Bravais group*; cf. [OPS98].

(iii) Up to isomorphism there are three types of real simple algebras with a definite involution, namely $(\mathbf{R}^{n\times n}, {}^{tr})$, $(\mathbf{C}^{n\times n}, {}^{-tr})$, and $(\mathbf{H}^{n\times n}, {}^{-tr})$, where $^-$ denotes complex, resp. quaternionic, conjugation. A (right) module for such $K^{n\times n}$ can be taken to be $K^{s\times n}$ with endomorphism ring $K^{s\times s}$ according to the three possibilities for K above. Then the $\mathbf{R}$-space of covariant forms can also be represented by $K^{s\times s}$, where the symmetric forms correspond to the symmetric matrices in case $K = \mathbf{R}$ and to the Hermitian matrices in the remaining two cases. According to the decomposition of $\mathcal{A}_{\mathbf{R}}$ into such simple components, one clearly has a decomposition of $\mathrm{Bil}_{\mathcal{A}_{\mathbf{R}}}(L_{\mathbf{R}})$ into components, each of which can be described as such a $K^{s \times s}$ with suitable K and s as above. In particular, this gives the $\mathbf{Z}$-ranks of $\mathrm{Bil}_\Lambda(L)$, $\mathrm{Bil}^+_\Lambda(L)$, and $\mathrm{Bil}^-_\Lambda(L)$.

It is often helpful to identify $\mathrm{Bil}_\Lambda(L) = \mathrm{Bil}_{\Lambda(L)}(L)$ with $\mathrm{Hom}_\Lambda(L, L^*) = \mathrm{Hom}_{\Lambda(L)}(L, L^*)$ as $\mathbf{Z}$-lattices. More precisely $\phi \in \mathrm{Bil}_\Lambda(L)$ is identified with $\tau \in \mathrm{Hom}_{\Lambda(L)}(L, L^*)$ by $W(\tau(V)) := \phi(V, W)$ for all $V, W \in L$, where we write $\Lambda(L)$-homomorphisms of right $\Lambda(L)$-modules on the left. As $\Lambda(L)$ can be replaced by any suborder Λ of $\Lambda(L)$ of finite index (invariant under the involution), we shall usually write Λ instead of $\Lambda(L)$ in the sequel. In this way, $\mathrm{Bil}_\Lambda(L^*)$ is also identified with $\mathrm{Hom}_\Lambda(L^*, L)$ and one gets bilinear maps $\mathrm{Bil}_\Lambda(L) \times \mathrm{Bil}_\Lambda(L^*) \to \mathrm{End}_\Lambda(L^*)$ and $\mathrm{Bil}_\Lambda(L^*) \times \mathrm{Bil}_\Lambda(L) \to \mathrm{End}_\Lambda(L)$, which can be composed with the reduced traces of the endomorphism rings of $L^*_{\mathbf{Q}}$ and of $L_{\mathbf{Q}}$ respectively, to obtain $\mathbf{Z}$-valued bilinear maps. Of course the latter become nondegenerate pairings if one tensors with the field of rational numbers. Hence one gets a discriminant for $\mathrm{Bil}_\Lambda(L)$, which measures the deviation of $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*))$ from being in perfect duality. Obviously, the same can be done for $\mathrm{Bil}^+_\Lambda(L)$ and $\mathrm{Bil}^-_\Lambda(L)$.

DEFINITION 2.3. Let ϵ stand for the empty symbol, $+$, or $-$. The *discriminant* of the pair $(\mathrm{Bil}^\epsilon_\Lambda(L), \mathrm{Bil}^\epsilon_\Lambda(L^*))$ is defined as

$$\mathrm{discr}\left(\mathrm{Bil}^\epsilon_\Lambda(L), \mathrm{Bil}^\epsilon_\Lambda(L^*)\right) := \left|\det\left(\mathrm{Tr}(\phi_i \psi_j)\right)_{1 \le i,j \le d}\right|,$$

where $(\phi_1, \ldots, \phi_d)$, resp. $(\psi_1, \ldots, \psi_d)$, form $\mathbf{Z}$-bases of $\mathrm{Bil}^\epsilon_\Lambda(L)$, resp. $\mathrm{Bil}^\epsilon_\Lambda(L^*)$, and Tr denotes the reduced trace of $\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$.

Clearly, the definitions are independent of the choice of bases and one can even define a discriminant group, whose order is the discriminant. As an easy exercise the reader may check that in the case of Example 2.2 (i) the discriminant $\mathrm{discr}(\mathrm{Bil}^+_\Lambda(L)), \mathrm{Bil}^+_\Lambda(L^*))$ is equal to the exponent of the discriminant group $L^{\sharp, f_0}/L$ of (L, ϕ_0), where $L^{\sharp, f_0} := \{V \in \mathcal{V} \mid \phi_0(L, V) \subseteq \mathbf{Z}\}$ with ϕ_0 the bilinear form described by f_0.

Another observation along the lines of the interplay between $\mathrm{Bil}_\Lambda(L)$, $\mathrm{Bil}_\Lambda(L^*)$, $\mathrm{End}_\Lambda(L)$, and $\mathrm{End}_\Lambda(L^*)$ is the presence of all of these in $\mathrm{End}_\Lambda(L \oplus L^*)$.

REMARK 2.4.

$$\mathrm{End}_{\mathcal{A}}(\mathcal{V}\oplus\mathcal{V}^*) = \begin{pmatrix} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) & \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*,\mathcal{V}) \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*) & \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*) \end{pmatrix}$$

is a $\mathbf{Q}$-algebra with involution $\begin{pmatrix} \zeta & \psi \\ \phi & \eta \end{pmatrix} \mapsto \begin{pmatrix} \eta^{tr} & \psi^{tr} \\ \phi^{tr} & \zeta^{tr} \end{pmatrix}$ and a C_2-graduation

$$\begin{pmatrix} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) & 0 \\ 0 & \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*) \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*,\mathcal{V}) \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*) & 0 \end{pmatrix}.$$

The involution is induced by the symmetric bilinear form ν on $\mathcal{V}\oplus\mathcal{V}^*$ defined by

$$\nu\colon (\mathcal{V}\oplus\mathcal{V}^*)\times(\mathcal{V}\oplus\mathcal{V}^*) \to \mathbf{Q} : \big((V_1,\varphi_1),(V_2,\varphi_2)\big) \mapsto V_1\varphi_2 + V_2\varphi_1$$

With respect to this bilinear form one has $(X\oplus Y)^{\#} = Y^*\oplus X^*$ for any two full lattices $X\subset\mathcal{V}$ and $Y\subset\mathcal{V}^*$. In particular, $\mathrm{End}_{\Lambda}(L\oplus L^*)$ is invariant under the involution.

The following proposition yields a better understanding of the discriminant.

PROPOSITION 2.5.

$$\mathrm{discr}\,\big(\mathrm{Bil}_{\Lambda}(L),\mathrm{Bil}_{\Lambda}(L^*)\big)\cdot \mathrm{discr}\,\big(\mathrm{End}_{\Lambda}(L)\big)^2 = \big|\mathrm{discr}\,\big(\mathrm{End}_{\Lambda}(L\oplus L^*)\big)\big|,$$

where the discriminants are taken with respect to the reduced traces.

Proof. One has $\mathrm{End}_{\Lambda}(L\oplus L^*) =$

$$\begin{pmatrix} \mathrm{End}_{\Lambda}(L) & 0 \\ 0 & \mathrm{End}_{\Lambda}(L^*) \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathrm{Hom}_{\Lambda}(L^*,L) \\ \mathrm{Hom}_{\Lambda}(L,L^*) & 0 \end{pmatrix}.$$

Since the two summands are orthogonal to each other with respect to the trace bilinear form, and since $\mathrm{End}_{\Lambda}(L)$ and $\mathrm{End}_{\Lambda}(L^*)$ are antiisomorphic and therefore have the same discriminant, the claim follows. □

Now the basic definition is well motivated.

DEFINITION 2.6. Let $(\mathcal{B},{}^\circ)$ be a $\mathbf{Q}$-algebra with a positive involution, and $\mathcal{W}$ a (faithful) $\mathcal{B}$-module containing a full $\mathbf{Z}$-lattice M. Let Γ be some suborder of finite index in $\Lambda(M)$. Finally let R be some subring of $\mathbf{R}$ containing $\mathbf{Z}$. We say that $\mathrm{Bil}_{\Lambda}(L)$ and $\mathrm{Bil}_{\Gamma}(M)$ are *R-equivalent* if there exists an R-module isomorphism $\omega\colon \mathrm{Bil}_{\Lambda_R}(L_R)\to\mathrm{Bil}_{\Lambda_R}(M_R)$, called an *$R$-equivalence*, which extends to an isomorphism Ω of R-algebras with involution and C_2-grading from $\mathrm{End}_{\Lambda_R}((L\oplus L^*)_R)$ onto $\mathrm{End}_{\Gamma_R}((M\oplus M^*)_R)$ and which induces a bijection from $\mathrm{Bil}^+_{\Lambda_R,>0}(L_R)$ onto $\mathrm{Bil}^+_{\Gamma_R,>0}(M_R)$. If $R=\mathbf{Z}$ then one simply says *equivalence* instead of $\mathbf{Z}$-equivalence.

It is worthwhile to spell out the isomorphism of $\mathrm{End}_{\Lambda_R}((L \oplus L^*)_R)$ onto $\mathrm{End}_{\Gamma_R}((M \oplus M^*)_R)$ in more detail. The equivalence ω: $\mathrm{Bil}_\Lambda(L_R) \to \mathrm{Bil}_{\Gamma_R}(M_R)$ obviously induces an R-module isomorphism ω': $\mathrm{Bil}_{\Lambda_R}(L_R^*) \to \mathrm{Bil}_{\Gamma_R}(M_R^*)$, for one may assume $R = \mathbf{R}$ and $\mathrm{Bil}_\Lambda(L_\mathbf{R})$ is spanned by nondegenerate (resp. invertible) elements ψ, and accordingly $\mathrm{Bil}_{\Lambda_\mathbf{R}}(L_\mathbf{R}^*)$ by the ψ^{-1}. The relation $\psi^{-1}\psi = id_{L_\mathbf{R}}$ translates into $\omega'(\psi^{-1}) = (\omega(\psi))^{-1}$. Obviously ω and ω', taken together, yield unique R-algebra isomorphisms ω_1: $\mathrm{End}_{\Lambda_R}(L_R) \to \mathrm{End}_{\Gamma_R}(M_R)$ and ω_2: $\mathrm{End}_{\Lambda_R}(L_R^*) \to \mathrm{End}_{\Gamma_R}((M_R^*)$, which are related by $\omega_2(\eta) = (\omega_1(\eta^{tr}))^{tr}$ for all $\eta \in \mathrm{End}_{\Gamma_R}((L_R^*)$. So one has the following

REMARK 2.7. In Definition 2.6 the R-algebra isomorphism

$$\Omega\colon \mathrm{End}_{\Lambda_R}((L \oplus L^*)_R) \to \mathrm{End}_{\Gamma_R}((M \oplus M^*)_R)$$

is uniquely determined by the equivalence ω: $\mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(M_R)$.

Obviously the discriminant of the pair $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*))$ and the discriminant group of $\mathrm{Bil}_\Lambda(L)$ do not change when one passes to an equivalent lattice of covariant forms. In the case of one-dimensional spaces of compatible forms, the discriminant separates equivalence classes.

REMARK 2.8. In the situation of Definition 2.6 let $\mathrm{Bil}_\mathcal{A}(\mathcal{V})$ and $\mathrm{Bil}_\mathcal{B}(\mathcal{W})$ be both one-dimensional. Then $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Lambda(M)$ are equivalent if and only if

$$\mathrm{discr}\,(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*)) = \mathrm{discr}\,(\mathrm{Bil}_\Gamma(M), \mathrm{Bil}_\Gamma(M^*))\,.$$

Proof. The missing direction follows from the following description of $\mathrm{End}_\Lambda(L \oplus L^*)$. Let $d := \mathrm{discr}\,(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}_\Lambda(L^*))$ and $\mathrm{Bil}_\Lambda(L) = \mathbf{Z}\phi$. Then

$$\mathrm{End}_\Lambda(L \oplus L^*) = \begin{pmatrix} \mathbf{Z}id_L & \mathbf{Z}d\phi^{-1} \\ \mathbf{Z}\phi & \mathbf{Z}id_{L^*} \end{pmatrix} \cong \begin{pmatrix} \mathbf{Z} & \mathbf{Z}d \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}. \qquad \square$$

From the discussion in Example 2.2 and the definition, it is reasonably clear that $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Lambda(M)$ are $\mathbf{R}$-equivalent if and only if $\mathrm{End}_{\mathcal{A}_\mathbf{R}}(\mathcal{V}_\mathbf{R})$ and $\mathrm{End}_{\mathcal{B}_\mathbf{R}}(\mathcal{W}_\mathbf{R})$ are isomorphic. For $\mathbf{Q}$-equivalence the statement is more difficult to prove.

PROPOSITION 2.9. *Let $(\mathcal{B}, {}^\circ)$ be a $\mathbf{Q}$-algebra with a positive involution, $\mathcal{W}$ a faithful $\mathcal{B}$-module containing a full $\mathbf{Z}$-lattice M, and let $\Gamma := \Lambda(M)$. Then two lattices $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Gamma(M)$ of covariant forms are $\mathbf{Q}$-equivalent if and only if $\mathrm{End}_\mathcal{A}(\mathcal{V})$ and $\mathrm{End}_\mathcal{B}(\mathcal{W})$ are isomorphic as $\mathbf{Q}$-algebras.*

Proof. By Definition 2.6 equivalence of $\mathcal{V}$ and $\mathcal{W}$ implies that the endomorphism rings are isomorphic. To prove the converse, we may assume without loss of generality that $\mathcal{A}$ and $\mathcal{B}$ are simple. Then the endomorphism rings are also simple. Fix an isomorphism $\omega_1\colon \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \to \mathrm{End}_{\mathcal{B}}(\mathcal{W})$. Then $\omega_2\colon \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*) \to \mathrm{End}_{\mathcal{B}}(\mathcal{W}^*) : \eta \mapsto (\omega_1(\eta^{tr}))^{tr}$ is also an isomorphism and $(\omega_2(\zeta^{tr}))^{tr} = \omega_1(\zeta)$ for all $\zeta \in \mathrm{End}_{\mathcal{A}}(\mathcal{V})$. To shorten the notation, set $\mathcal{E} := \mathrm{End}_{\mathcal{A}}(\mathcal{V})$ and $\mathcal{E}' := \mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$. (Note, transposing induces an antiisomorphism between $\mathcal{E}$ and $\mathcal{E}'$.)

The next aim is to find a suitable map

$$\omega\colon \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{W}, \mathcal{W}^*)$$

as required in Definition 2.6. Clearly, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$ is a simple $(\mathcal{E}', \mathcal{E})$-bimodule. The two isomorphisms ω_1 and ω_2 can be used to turn $\mathrm{Hom}_{\mathcal{B}}(\mathcal{W}, \mathcal{W}^*)$ into a simple $(\mathcal{E}', \mathcal{E})$-bimodule as well. Then ω lies in

$$H := \mathrm{Hom}_{(\mathcal{E}', \mathcal{E})}(\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*), \mathrm{Hom}_{\mathcal{B}}(\mathcal{W}, \mathcal{W}^*)),$$

which is a one-dimensional Z-module, where Z is the centre of $\mathcal{E}$. To get the right identification of Z with the centre of $\mathcal{E}'$, note that the centres of $\mathcal{A}$ and $\mathcal{E}$ can be identified via their action on $\mathcal{V}$ and that $z \mapsto (z^\circ)^{tr}$ therefore gives the right identification of Z with $Z(\mathcal{E}')$.

Now some properties of H have to be investigated: For each $h \in H$ define h^{tr} by $h^{tr}(\phi) := h(\phi^{tr})^{tr}$ for all $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*, \mathcal{V})$. This defines a Z-semilinear action of the cyclic group of order 2 on H. Indeed, one easily checks: $(h^{tr})^{tr} = h$ and $(zh)^{tr} = z^\circ h^{tr}$ for all $z \in Z$ and all $h \in H$. Next, one verifies that there exists a nonzero $h \in H$ with $h^{tr} = h$. Indeed, if $^\circ$ fixes Z pointwise, any $h \in H$ is fixed by ^{tr}, because the subspaces of symmetric and skewsymmetric forms have different dimensions in this case. If $^\circ$ does not fix Z pointwise, the existence of an $h \in H$ with $h^{tr} = h$ follows from a straightforward analysis of semilinear C_2-actions. In any case, the $h \in H$ with $h^{tr} = h$ form a one-dimensional $\widetilde{Z}$-subspace $\widetilde{H}$ of H, where $\widetilde{Z}$ is the $^\circ$-fixed subfield of Z. It is clear that any symmetric $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*) = \mathrm{Bil}_{\Lambda}(\mathcal{V})$ is mapped onto a symmetric $\omega(\phi)$ by any $\omega \in \widetilde{H}$. The final point is that ω can be chosen in such a way that positive definite forms map onto positive definite ones. This can easily be seen for the ground field $\mathbf{R}$ by the classification of the simple $\mathbf{R}$-algebras with positive involutions. The present case of rational ground field can be reduced to the previous case, i.e. if $h \in \widetilde{H}$ does not respect positive definite forms, then there exists a $z \in \widetilde{Z}$ with the right sign combinations in the various archimedean completions of $\widetilde{Z}$ such that zh maps positive forms onto positive ones. One ends up with a nonzero

$\omega \in \widetilde{H}$ respecting positiveness, which is unique up to multiplication with totally positive elements in $\widetilde{Z}$.

Similarly one finds a suitable map ω': $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*, \mathcal{V}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{W}^*, \mathcal{W})$ as required in Definition 2.6. Finally, to make ω' unique, one requires $\omega'(\phi^{-1}) = \omega(\phi)^{-1}$ for one (and hence for all) invertible $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$. Now it is a routine matter to check that $(\omega_1, \omega_2, \omega, \omega')$ defines an algebra isomorphism Ω of $\mathrm{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*)$ onto $\mathrm{End}_{\mathcal{B}}(\mathcal{W} \oplus \mathcal{W}^*)$ with the required properties. □

At the end of this basic chapter some comments might be in place: The reader should check as a little exercise that $\mathrm{Bil}_\Lambda(L)$ (given as explicit bilinear forms or as maps from L to L^*) determines $\mathrm{End}_\Lambda(L)$ (but not conversely of course) and $\mathrm{End}_\Lambda(L \oplus L^*)$. One now may ask how much is determined by $\mathrm{Bil}_\Lambda^+(L)$.

DEFINITION 2.10. Call L, $\mathcal{V}$ and $\mathrm{Bil}_\Lambda(L)$ *exceptional*, if $\mathrm{End}_{\mathcal{A}_{\mathbf{R}}}(\mathcal{V}_{\mathbf{R}})$ has a simple component isomorphic to $\mathbf{C}$ or $\mathbf{H}$.

REMARK 2.11. The following three conditions are equivalent.

(i) $\mathrm{Bil}_\Lambda^-(L)$ can be recovered from $\mathrm{Bil}_\Lambda^+(L)$;

(ii) $\mathrm{End}_\Lambda(L)$ can be recovered from $\mathrm{Bil}_\Lambda^+(L)$;

(iii) L is not exceptional.

For instance the difference between the Bravais group and the strict Bravais group in Example 2.2 (ii) only occurs in the exceptional situation.

3. AUTOEQUIVALENCES AND INVARIANTS

The basic notation is kept: $(\mathcal{A}, ^\circ)$, $L \subset \mathcal{V}$, $\mathrm{Bil}_\Lambda(L) \equiv \mathrm{Hom}_\Lambda(L, L^*)$. Continuing Definition 2.6 in the direction 'autoequivalences', we fix the following notation.

DEFINITION 3.1. Let R be a subring of $\mathbf{R}$ containing $\mathbf{Z}$. The group of all R-equivalences ω: $\mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_\Lambda(L_R)$ is denoted by $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))$.