Zeitschrift: L'Enseignement Mathématique

Band: 47 (2001)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LATTICES OF COVARIANT QUADRATIC FORMS

Kapitel: 3. Autoequivalences and invariants

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DOI: https://doi.org/10.5169/seals-65427

$\omega \in \widetilde{H}$ respecting positiveness, which is unique up to multiplication with totally positive elements in $\widetilde{Z}$.

Similarly one finds a suitable map ω': $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}^*, \mathcal{V}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{W}^*, \mathcal{W})$ as required in Definition 2.6. Finally, to make ω' unique, one requires $\omega'(\phi^{-1}) = \omega(\phi)^{-1}$ for one (and hence for all) invertible $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$. Now it is a routine matter to check that $(\omega_1, \omega_2, \omega, \omega')$ defines an algebra isomorphism Ω of $\mathrm{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*)$ onto $\mathrm{End}_{\mathcal{B}}(\mathcal{W} \oplus \mathcal{W}^*)$ with the required properties. $\square$

At the end of this basic chapter some comments might be in place: The reader should check as a little exercise that $\mathrm{Bil}_\Lambda(L)$ (given as explicit bilinear forms or as maps from L to L^*) determines $\mathrm{End}_\Lambda(L)$ (but not conversely of course) and $\mathrm{End}_\Lambda(L \oplus L^*)$. One now may ask how much is determined by $\mathrm{Bil}_\Lambda^+(L)$.

DEFINITION 2.10. Call L, $\mathcal{V}$ and $\mathrm{Bil}_\Lambda(L)$ *exceptional*, if $\mathrm{End}_{\mathcal{A}_\mathbf{R}}(\mathcal{V}_\mathbf{R})$ has a simple component isomorphic to $\mathbf{C}$ or $\mathbf{H}$.

REMARK 2.11. The following three conditions are equivalent.

(i) $\mathrm{Bil}_\Lambda^-(L)$ can be recovered from $\mathrm{Bil}_\Lambda^+(L)$;

(ii) $\mathrm{End}_\Lambda(L)$ can be recovered from $\mathrm{Bil}_\Lambda^+(L)$;

(iii) L is not exceptional.

For instance the difference between the Bravais group and the strict Bravais group in Example 2.2 (ii) only occurs in the exceptional situation.

3. AUTOEQUIVALENCES AND INVARIANTS

The basic notation is kept: $(\mathcal{A}, {}^\circ)$, $L \subset \mathcal{V}$, $\mathrm{Bil}_\Lambda(L) \equiv \mathrm{Hom}_\Lambda(L, L^*)$. Continuing Definition 2.6 in the direction 'autoequivalences', we fix the following notation.

DEFINITION 3.1. Let R be a subring of $\mathbf{R}$ containing $\mathbf{Z}$. The group of all R-equivalences ω: $\mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_\Lambda(L_R)$ is denoted by $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))$.

From the discussion following Definition 2.6, it is clear that one has a monomorphism of $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L_R))$ into the group of all automorphisms of $\mathrm{End}_{\Lambda_R}((L \oplus L^*)_R)$ as a C_2-graded algebra with involution, and also into the automorphism group of $\mathrm{End}_{\Lambda_R}(L_R)$. It therefore makes sense to look at the pointwise stabilizer of the centre of $\mathrm{End}_{\Lambda_R}(L_R)$.

REMARK 3.2. Denote by $\mathrm{Aut}^e_z(\mathrm{Bil}_{\Lambda_R}(L_R))$ the biggest subgroup of $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L_R))$ fixing the centre of $\mathrm{End}_{\Lambda_R}(L_R)$

(i) $\mathrm{Aut}^e_z(\mathrm{Bil}_{\Lambda_R}(L_R))$ is a normal subgroup of $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))$ of finite index with the factor group $\mathrm{Aut}^e(\mathrm{Bil}_{\Lambda_R}(L_R))/\mathrm{Aut}^e_z(\mathrm{Bil}_{\Lambda_R}(L_R))$ acting faithfully on the centre of $\mathrm{End}_{\Lambda_R}(L_R)$.

(ii) $\mathrm{Aut}^e_z(\mathrm{Bil}_{\Lambda_R}(L_R))$ is isomorphic to the group of inner automorphisms of $\mathrm{End}_{\Lambda_R}(L_R)$ in case R is a field.

(iii) If R is not a field, let Q be its field of fractions. Then $\mathrm{Aut}^e_z(\mathrm{Bil}_{\Lambda_R}(L_R))$ is isomorphic to a subgroup of $\mathrm{Aut}^e_z(\mathrm{Bil}_{\Lambda_Q}(L_Q))$.

Proof. (i) Finite dimensional semisimple commutative algebras have finite automorphism groups. The same applies to R-orders in such algebras.

(ii) This follows from the Skolem-Noether Theorem.

(iii) Obvious. □

PROPOSITION 3.3. *The group* $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ *acts properly discontinuously on* $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(L_{\mathbf{R}})$.

Proof. That $\mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ acts on $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(L_{\mathbf{R}})$ follows from the definition of equivalence. By Remark 3.2 it suffices to show that $\mathrm{Aut}^e_z(\mathrm{Bil}_\Lambda(L))$ acts properly discontinuously. But this follows from the well known fact that $\mathrm{GL}_n(\mathbf{Z})$ acts properly discontinuously on the cone of positive definite symmetric matrices of degree n. □

In fact, the action is even discontinuous on $\mathrm{Bil}^+_{\Lambda_{\mathbf{R}},>0}(L_{\mathbf{R}})$ modulo the action of $\mathbf{R}_{>0}$ by multiplication and, apart from some marginal exceptions, it is also faithful. One interesting issue is the structure and size of $\mathrm{Out}^e_z(\mathrm{Bil}_\Lambda(L))$, to be defined now.

DEFINITION 3.4.

(i) The subgroup of $\mathrm{Aut}^e_z(\mathrm{Bil}_\Lambda(L))$ corresponding to the inner automorphisms of $\mathrm{End}_\Lambda(L)$ will be denoted by $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$ and referred to as *the group of inner automorphisms of* $\mathrm{Bil}_\Lambda(L)$. (Clearly $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L)) \cong \mathrm{Inn}(\mathrm{End}_\Lambda(L)) \cong (\mathrm{End}_\Lambda(L))^*/Z(\mathrm{End}_\Lambda(L)^*)$.)

(ii) Similarly,

$$\mathrm{Out}^e_z(\mathrm{Bil}_\Lambda(L)) := \mathrm{Aut}^e_z(\mathrm{Bil}_\Lambda(L))/\,\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$$

will be *the outer central group of equivalences of* $\mathrm{Bil}_\Lambda(L)$, and

$$\mathrm{Out}^e(\mathrm{Bil}_\Lambda(L)) := \mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))/\,\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$$

the outer group of equivalences of $\mathrm{Bil}_\Lambda(L)$.

$$\mathrm{Out}^e(\mathrm{Bil}(L))\left\{\begin{array}{l}\bullet\ \mathrm{Aut}^e(\mathrm{Bil}(L))\\ \left.\begin{array}{l}\bullet\ \mathrm{Aut}^e_z(\mathrm{Bil}(L))\\ \bullet\ \mathrm{Inn}(\mathrm{Bil}(L))\end{array}\right\}\mathrm{Out}^e_z(\mathrm{Bil}(L))\end{array}\right.$$

PROPOSITION 3.5. *The group* $\mathrm{Out}^e(\mathrm{Bil}_\Lambda(L))$ *is well defined and embeds into* $\mathrm{Out}(\mathrm{End}_\Lambda(L)) := \mathrm{Aut}(\mathrm{End}_\Lambda(L))/\,\mathrm{Inn}(\mathrm{End}_\Lambda(L))$. *In particular, it is finite.*

Proof. Clearly, conjugation by $\omega \in \mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ of an inner automorphism induced by some $\varphi \in \mathrm{End}_\Lambda(L)^*$ results in the inner automorphism induced by $\omega_1(\varphi)$ in the notation of the discussion of Definition 2.6. Hence $\mathrm{Out}^e(\mathrm{Bil}_\Lambda(L))$ is well defined. The finiteness follows from the Jordan-Zassenhaus Theorem, which implies that $\mathrm{Out}(\Gamma)$ is finite for any $\mathbf{Z}$-order Γ in a semisimple $\mathbf{Q}$-algebra, cf. [CuR87] (55.19). □

Obviously $\mathrm{Out}^e_z(\mathrm{Bil}_\Lambda(L))$ is an interesting invariant for the equivalence class of $\mathrm{Bil}_\Lambda(L)$. Further on in this chapter, it will be proved that it is an Abelian group in case $\mathrm{End}_\Lambda(L)$ is hereditary. But some notions from the theory of orders first have to be recalled, in order to define some invariants measuring the distance from this favourable situation.

Recall from [BeZ85] that the *arithmetical radical* $\mathrm{arad}(\Gamma)$ of a $\mathbf{Z}$-order Γ in a semisimple $\mathbf{Q}$-algebra $\mathcal{B}$ is defined as the ideal which localizes to the radical of Γ_p at the primes dividing the discriminant of Γ, and to the localization Γ_p of the order itself at the other primes. The *left idealizer* or *left order* $\Gamma^{(l)}$ of the arithmetical radical $\mathrm{arad}(\Gamma)$ is the biggest $\mathbf{Z}$-order in $\mathcal{B}$ in which $\mathrm{arad}(\Gamma)$ is a left ideal, in particular $\Gamma^{(l)}\,\mathrm{arad}(\Gamma) \subseteq \mathrm{arad}(\Gamma)$. It is well known, cf. [Rei75], that Γ is hereditary if and only if $\Gamma = \Gamma^{(l)}$. Likewise the two-sided idealizer of $\mathrm{arad}(\Gamma)$ is the biggest $\mathbf{Z}$-order in $\mathcal{B}$ having $\mathrm{arad}(\Gamma)$ as a two-sided ideal. It is denoted by $\Gamma^{(rl)}$. A slight modification of the argument in [Rei75] characterizing hereditary orders by the property $\Gamma = \Gamma^{(l)}$ also shows that Γ is hereditary if and only if $\Gamma = \Gamma^{(rl)}$. Besides, if Γ is invariant under an involution of $\mathcal{B}$, so is $\Gamma^{(rl)}$. Define the left, respectively two-sided, *idealizer* *sequence* of Γ by $\Gamma_0 := \Gamma$ and $\Gamma_{i+1} := \Gamma_i^{(l)}$, resp. $\Gamma_{i+1} := \Gamma_i^{(rl)}$, for $i \geq 0$. The *length* of either of these sequences is the smallest i with $\Gamma_i = \Gamma_{i+1}$.

DEFINITION 3.6.

(i) The *e-depth* of $\mathrm{Bil}_\Lambda(L)$ is defined as the length of the left idealizer sequence of $\mathrm{End}_\Lambda(L)$.

(ii) The *e-$*$-depth* of $\mathrm{Bil}_\Lambda(L)$ is defined as the length of the two-sided idealizer sequence of $\mathrm{End}_\Lambda(L \oplus L^*)$.

Clearly, e-depth and e-$*$-depth are well defined and compatible with equivalence. As for the definition of e-$*$-depth, note that all members of the two-sided idealizer sequence of $\mathrm{End}_\Lambda(L \oplus L^*)$ are both C_2-graded and invariant under the involution of $\mathrm{End}_{\mathcal{A}}(\mathcal{V} \oplus \mathcal{V}^*)$. However, it does not seem that they are necessarily endomorphism rings of lattices $M \oplus M^*$ with the M's constructed from L in a canonical way. That is why we shall focus here mainly on the e-$*$-depth, resp. e-depth, zero case. The general discussion will be resumed in the next section; cf. 4.8 and 4.10. Already the case of one-dimensional $\mathrm{Bil}_\Lambda(L)$ shows that even if the e-depth is zero, the e-$*$-depth can be arbitrarily large, since the discriminant of $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}(L^*))$ can be arbitrarily big. However, it seems that for every isomorphism type of $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ the equivalence classes of e-$*$-depth 0 lattices $\mathrm{Bil}_\Lambda(L)$ of covariant integral forms can be classified, provided one restricts the number of primes involved in the discriminant. Here is an example, whose verification is left to the reader as an exercise in combinatorics.

EXAMPLE 3.7. Let $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$. Assume that $\mathrm{Bil}_\Lambda(L)$ is of e-$*$-depth zero and that the discriminant of the pair $(\mathrm{Bil}_\Lambda(L), \mathrm{Bil}(L^*))$ is a power of a prime p. Then there are nine equivalence classes of such lattices and the endomorphism rings satisfy $\mathrm{End}_\Lambda(L \oplus L^*) \cong X(E)$ with E one of the matrices

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&1&1\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&1&1\\0&0&0&1\\0&-1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1\\1&0&1&1\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\1&0&1&1\\0&0&0&1\\0&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\1&0&1&2\\0&0&0&1\\0&-1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1\\1&0&1&2\\0&-1&0&1\\-1&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&2\\1&0&2&2\\0&-1&0&1\\-1&-1&0&0 \end{pmatrix}$$

and $X((m_{ij})) := \{(x_{ij}) \in \mathbf{Q}^{n\times n} \mid x_{ij} \in p^{m_{ij}}\mathbf{Z}\}$ for any $(m_{ij}) \in \mathbf{Z}^{n\times n}$.

In all cases $L \cong L_1 \oplus L_2$ with irreducible Λ-lattices L_1, L_2 satisfying $pL_1 \le L_2 \le L_1 \le L_1^{\#} \le p^{-1}L_1$, where the dual lattice is taken with respect to the positive definite generator of $\mathrm{Bil}_\Lambda(L_1)$. The individual cases are characterized by a chain of inclusions which can be read off from the rows of the matrices, like $L_1 = L_2 = L_1^{\#} = L_2^{\#}$ in the first case, $L_1 = L_2 < L_1^{\#} = L_2^{\#}$ in the second case, $pL_2^{\#} < L_2 < L_1 = L_1^{\#} < L_2^{\#}$ in the third case, or $pL_2^{\#} = L_2 < L_1 = L_1^{\#} < L_2^{\#}$ in the fourth case. Moreover, the second, the fourth, and the last three cases might have outer automorphisms.

To proceed to the promised structure theorem on $\mathrm{Out}_z^e(\mathrm{Bil}_\Lambda(L)$ for the depth 0 case, the following lemma is needed, which is implicit in [Neb98] and which certainly does not depend on the big Picard group machinery of [CuR87], Chapter 55.

LEMMA 3.8. *Let Γ be a hereditary order in a simple $\mathbf{Q}$-algebra $\mathcal{B}$, which has Schur index s and degree d (over its centre). Then $\mathrm{Out}_z(\Gamma)$ is Abelian. Moreover, if n is the number of primes in the centre $Z(\Gamma)$ dividing the discriminant of Γ with respect to the centre, then $\mathrm{Out}_z(\Gamma)$ can be embedded into an extension of the class group $Cl(Z(\Gamma))$ by $(C_{sd})^n$.*

Proof. Define $N(\Gamma) := \{b \in \mathcal{B}^* \mid b\Gamma b^{-1} = \Gamma\}$. Then $\mathrm{Out}_z(\Gamma) \cong N(\Gamma)/\langle\Gamma^*, Z(\mathcal{B})^*\rangle$. Let $\mathcal{W}$ be an irreducible $\mathcal{B}$-module. Then $N(\Gamma)$ acts on the Γ-sublattices in $\mathcal{W}$. For every prime $\mathfrak{p}$ in the centre of Γ, the $\Gamma_\mathfrak{p}$-sublattices in the completion $\mathcal{W}_\mathfrak{p}$ form a chain by inclusion, on which $N(\Gamma)$ acts by shifting the lattices up and down. Clearly the intersection of all the kernels of these shifts at the various primes is Γ^*. Hence $\mathrm{Out}_z(\Gamma)$ is Abelian.

More precisely, let $\mathrm{Sh}(\mathcal{W})$ be the group of all permutations of the Γ-sublattices of $\mathcal{W}$ which fixes all lattices in $\mathcal{W}_\mathfrak{p}$ for almost all primes $\mathfrak{p}$ in the centre of Γ and induces shifts at the remaining finitely many completions. Then $\mathrm{Sh}(\mathcal{W})$ is the direct sum of the $\mathrm{Sh}(\mathcal{W}_\mathfrak{p})$, each of which is infinite cyclic. Moreover $\mathrm{Sh}(\mathcal{W})$ acts regularly on the set of all nonzero Γ-lattices in $\mathcal{W}$. The above argument shows that $N(\Gamma)/\Gamma^*$ embeds into $\mathrm{Sh}(\mathcal{W})$. But so does the group F of all fractional ideals of $Z(\mathcal{B})$, resulting in a subgroup $\overline{F}$ of $\mathrm{Sh}(\mathcal{W})$. The cokernel of this embedding is isomorphic to a subgroup of a direct product of n cyclic groups, the order of each one of which divides sd. It is well known that $Z(\mathcal{B})^*$ maps into F with cokernel the class group of $Z(\Gamma)$ and kernel the torsion subgroup of $Z(\mathcal{B})$, which lies in Γ^* anyhow. Now by the above description of $\mathrm{Out}(\Gamma)$, it can be viewed as a subgroup of

$\mathrm{Sh}(\mathcal{W})/X$, where X is the image of $Z(\mathcal{B})^*$ in $\mathrm{Sh}(\mathcal{W})$. But $\mathrm{Sh}(\mathcal{W})/X$ is also an extension of $F/X \cong Cl(Z(\Gamma))$ by $\mathrm{Sh}(\mathcal{W})/\bar{F}$. □

As a consequence one gets the following

THEOREM 3.9. *Let* $\mathrm{Bil}_\Lambda(L)$ *be of depth* 0, *then* $\mathrm{Out}_z^e(\mathrm{Bil}_\Lambda(L))$ *is Abelian.* □

It is worthwhile to extract more precise statements from Lemma 3.8. They will be used and extended in the forthcoming chapter in the study of $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ when the e-$*$-depth of $\mathrm{Bil}_\Lambda(L)$ is zero.

DEFINITION 3.10. Let Γ be a hereditary $\mathbf{Z}$-order in a simple $\mathbf{Q}$-algebra $\mathcal{B}$ and let $\mathfrak{p}$ be a prime ideal in the centre $Z(\Gamma)$ of Γ. The $\mathfrak{p}$-*local shift index* $s(\Gamma,\mathfrak{p})$ of Γ is defined as follows: For any irreducible $\Lambda_\mathfrak{p}$-lattice L define $m(L)$ by $\mathfrak{p}^{m(L)} := [L : L_{max}]$, where L_{max} is the unique maximal Γ-sublattice of L. The chain $\cdots L_i \geq L_{i+1} \cdots$ of irreducible lattices in a simple $\mathcal{B}_\mathfrak{p}$-module $\mathcal{W}$ yields a periodic sequence $\ldots, m(L_i), m(L_{i+1}), \ldots$ because of $m(L) = m(\mathfrak{p}L)$. The index of the group of all "central" shifts generated by multiplication with $\mathfrak{p}$ in the group of all shifts of the chain respecting $m(L)$ is called $s(\Gamma,\mathfrak{P})$.

Obviously, $s(\Gamma,\mathfrak{p})$ is equal to the $\mathfrak{p}$-local Schur index of $\mathcal{B}$ if $\Gamma_\mathfrak{p}$ is a maximal order. In particular it is almost always equal to 1. With the definition of the local shift index at hand, the refined statement of Lemma 3.8, which was actually proved, should read as stated with $(C_{sd})^n$ replaced by $\bigoplus_\mathfrak{p} C_{c(\Gamma,\mathfrak{p})}$.

4. EXTRINSIC NOTIONS: USING THE UNDERLYING LATTICE

Up to now, the lattices $\mathrm{Bil}_\Lambda(L)$ of covariant forms have only been investigated by themselves without much reference to the underlying lattice L. In this section L will be taken more seriously into account. Unless confusion can arise L will also denote the underlying $\mathbf{Z}$-lattice of L, which is usually considered as a Λ-lattice.

To start with, we discuss the determinant function and its behaviour under equivalence.