# 4. EXTRINSIC NOTIONS : USING THE UNDERLYING LATTICE

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

$\mathrm{Sh}(\mathcal{W})/X$, where $X$ is the image of $Z(\mathcal{B})^*$ in $\mathrm{Sh}(\mathcal{W})$. But $\mathrm{Sh}(\mathcal{W})/X$ is also an extension of $F/X \cong Cl(Z(\Gamma))$ by $\mathrm{Sh}(\mathcal{W})/\bar{F}$. □

As a consequence one gets the following

THEOREM 3.9. *Let* $\mathrm{Bil}_\Lambda(L)$ *be of depth* 0, *then* $\mathrm{Out}_z^e(\mathrm{Bil}_\Lambda(L))$ *is Abelian.* □

It is worthwhile to extract more precise statements from Lemma 3.8. They will be used and extended in the forthcoming chapter in the study of $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ when the e-$*$-depth of $\mathrm{Bil}_\Lambda(L)$ is zero.

DEFINITION 3.10. Let $\Gamma$ be a hereditary $\mathbf{Z}$-order in a simple $\mathbf{Q}$-algebra $\mathcal{B}$ and let $\mathfrak{p}$ be a prime ideal in the centre $Z(\Gamma)$ of $\Gamma$. The $\mathfrak{p}$-*local shift index* $s(\Gamma, \mathfrak{p})$ of $\Gamma$ is defined as follows: For any irreducible $\Lambda_\mathfrak{p}$-lattice $L$ define $m(L)$ by $\mathfrak{p}^{m(L)} := [L : L_{max}]$, where $L_{max}$ is the unique maximal $\Gamma$-sublattice of $L$. The chain $\cdots L_i \geq L_{i+1} \cdots$ of irreducible lattices in a simple $\mathcal{B}_\mathfrak{p}$-module $\mathcal{W}$ yields a periodic sequence $\ldots, m(L_i), m(L_{i+1}), \ldots$ because of $m(L) = m(\mathfrak{p}L)$. The index of the group of all "central" shifts generated by multiplication with $\mathfrak{p}$ in the group of all shifts of the chain respecting $m(L)$ is called $s(\Gamma, \mathfrak{P})$.

Obviously, $s(\Gamma, \mathfrak{p})$ is equal to the $\mathfrak{p}$-local Schur index of $\mathcal{B}$ if $\Gamma_\mathfrak{p}$ is a maximal order. In particular it is almost always equal to 1. With the definition of the local shift index at hand, the refined statement of Lemma 3.8, which was actually proved, should read as stated with $(C_{sd})^n$ replaced by $\bigoplus_\mathfrak{p} C_{c(\Gamma,\mathfrak{p})}$.

## 4. EXTRINSIC NOTIONS: USING THE UNDERLYING LATTICE

Up to now, the lattices $\mathrm{Bil}_\Lambda(L)$ of covariant forms have only been investigated by themselves without much reference to the underlying lattice $L$. In this section $L$ will be taken more seriously into account. Unless confusion can arise $L$ will also denote the underlying $\mathbf{Z}$-lattice of $L$, which is usually considered as a $\Lambda$-lattice.

To start with, we discuss the determinant function and its behaviour under equivalence.

DEFINITION 4.1.

$$\det\colon \mathrm{Bil}_\Lambda(L) \to \mathbf{Z} : \phi \mapsto \det(\phi_B)$$

is called the *determinant function* on $\mathrm{Bil}_\Lambda(L)$, where $B$ is some lattice basis for $L$ over $\mathbf{Z}$ and $\phi_B$ is the Gram matrix of $\phi$ with respect to $B$.

Clearly, choosing some $\mathbf{Z}$-basis for $\mathrm{Bil}_\Lambda(L)$ turns the determinant into a homogeneous polynomial in $\mathbf{Z}[X_1,\ldots,X_d]$ of degree $n = \dim_{\mathbf{Z}}(L)$ in $d = \dim_{\mathbf{Z}}(\mathrm{Bil}_\Lambda(L))$ variables. A connection of the factorization properties in $\mathbf{Q}[X_1,\ldots,X_d]$ with the structure of $\mathcal{V}$ is indicated in the rather obvious Remark 4.2 below. Those in $\mathbf{Z}[X_1,\ldots,X_d]$ have not yet been investigated. There sometimes seem to be changes in the factorization behaviour when one restricts from $\mathrm{Bil}_\Lambda(L)$ to $\mathrm{Bil}_\Lambda^+(L)$; cf. Chapter 5.

REMARK 4.2. Let $1 = e_1 + \ldots + e_h$ be the decomposition of $1 \in \mathcal{A}$ into central primitive idempotents of $\mathcal{A}$, and fix some isomorphism $\psi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{V},\mathcal{V}^*)$. There is a constant $a = a(\psi, L) \in \mathbf{Q}$ depending on $\psi$ such that, for all $\phi \in \mathrm{Bil}_\Lambda(L)$, one has

$$\det(\phi) = a \cdot \prod_{i=1}^{h} (\det\nolimits_{\mathrm{red}}(\psi_i\phi_i))^{m(i)},$$

where the $\phi_i$ and $\psi_i$ denote the restrictions of $\phi$, resp. $\psi$, to $\mathcal{V}e_i$, resp. to $\mathcal{V}^*e_i$, $\det_{\mathrm{red}}$ is the reduced determinant of $\mathrm{End}_{\mathcal{A}}(\mathcal{V}e_i)$, and finally $m(i)$ is the degree of the matrix algebras which are the simple components of $\mathbf{C} \otimes_{\mathbf{Q}} \mathcal{A}e_i$.

If $\omega\colon \mathrm{Bil}_\Lambda(L) \to \mathrm{Bil}_\Lambda(M)$ is an equivalence, only the constant $a$ in the above formula changes to some other constant $b = b(\omega'(\psi), M)$, and the exponents $m(i)$ change to the degrees $m(i)'$ of the corresponding simple components of $\mathbf{C} \otimes_{\mathbf{Q}} \mathcal{B}e_i'$. One has

$$\det(\omega(\phi)) = b \cdot \prod_{i=1}^{h} (\det\nolimits_{\mathrm{red}}(\psi_i\phi_i))^{m(i)'},$$

since $\det_{\mathrm{red}}(\psi_i\phi_i) = \det_{\mathrm{red}}(\omega'(\psi_i)\omega(\phi_i))$, cf. discussion of Definition 2.6.

As an instructive example, which comes up as a step in the proof of Remark 4.2, the reader may want to relate the above formula to the well known determinant formula for the Kronecker product of two matrices.

DEFINITION 4.3. Let $(\mathcal{B},^\circ)$, $\mathcal{W}$, $M$, and $R$ be as in Definition 2.6.

(i) Call $L$ and $M$ *form-$R$-equivalent*, or simply *form-equivalent* in case $R = \mathbf{Z}$, if there is an $R$-module isomorphism $\tau\colon M_R \to L_R$ which induces an $R$-equivalence $\omega\colon \mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(M_R) : \phi \mapsto \omega(\phi) = \tau\phi$ with $\tau\phi(W_1, W_2) = \phi(W_1\tau, W_2\tau)$ for all $W_1, W_2 \in M_R$. In this case $(L, \mathrm{Bil}_\Lambda(L))$ and $(M, \mathrm{Bil}_\Gamma(M))$ are also called $R$-equivalent and the $R$-equivalence $\omega$ is said to be *induced.*

(ii) We denote by $N(L_R)$ the group of all $\tau \in \mathrm{Aut}_R(L_R)$ inducing autoequivalences of $\mathrm{Bil}_\Lambda(L_R)$.

(iii) The group of all induced autoequivalences of $\mathrm{Bil}_{\Lambda_R}(L_R)$ is denoted $\mathrm{Aut}\big(\mathrm{Bil}_{\Lambda_R}(L_R)\big)$ ; its elements are also called *automorphisms of* $\mathrm{Bil}_\Lambda(L_R)$.

The connection with the earlier concepts is easily seen: for the determinant functions, one has $\det(\phi) = \det(\omega(\phi))$ for all $\phi \in \mathrm{Bil}_\Lambda(L)$ if the equivalence $\omega\colon \mathrm{Bil}_\Lambda(L) \to \mathrm{Bil}_\Gamma(M)$ is induced, i.e. the constant and the exponents in the formula of Remark 4.2 do not change any more. In other words, the associated polynomials in $\mathbf{Z}[X_1, \ldots, X_d]$ are $\mathbf{Z}$-equivalent, or even equal if one chooses appropriately the bases of the lattices of forms. Clearly, $\mathrm{Inn}(\mathrm{Bil}_\Lambda(L)) \leq \mathrm{Aut}(\mathrm{Bil}_\Lambda(L)) \leq \mathrm{Aut}^e(\mathrm{Bil}_\Lambda(L))$ with all indices finite.

To get a full picture of the situation, one more group has to be introduced, namely the kernel of the epimorphism of $N(L)$ onto $\mathrm{Aut}(\mathrm{Bil}_\Lambda(L))$, which is $U(L)$ defined as follows.

DEFINITION 4.4.

(i) $U(L_R)$ is the image of the group $U\big(\Lambda(L)_R\big) := \{u \in \Lambda(L)_R \mid uu^\circ = 1\}$ in $\mathrm{Aut}_R(L_R)$ defined by its natural linear action on $L_R$.

(ii) The exact sequence

$$1 \to U(L) \to N(L) \to \mathrm{Aut}(\mathrm{Bil}_\Lambda(L)) \to 1$$

is called the *basic exact sequence.*

Obviously $U(L)$ is finite. If the $\mathbf{Q}$-algebra spanned by $U(L)$ is all of the image $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathrm{End}_\mathbf{Q}(\mathcal{V})$, then $N(L)$ is the normalizer of (the strict Bravais group) $U(L)$ in $\mathrm{Aut}_\mathbf{Z}(L)$; cf. [BNZ73]. In general one only has that $N(\mathcal{V})$ is the normalizer of $U(\mathcal{V})$ in $\mathrm{Aut}_\mathbf{Q}(\mathcal{V})$. The structure of $N(\mathcal{V})$ is easily worked out: it is dominated by the pair of semisimple subalgebras $\overline{\mathcal{A}}$ and $\mathrm{End}_\mathcal{A}(\mathcal{V})$ of $\mathrm{End}_\mathbf{Q}(\mathcal{V})$, which are centralizers of each other. In fact, if one restricts to the pointwise stabilizer $N_z(\mathcal{V})$ of the common centre of these two algebras, then $N_z(\mathcal{V})$ is the central product of $\mathrm{End}_\mathcal{A}(\mathcal{V})^*$ and a group $\widetilde{U}(\mathcal{V})$ amalgamated over their common centre, where $\widetilde{U}(\mathcal{V})$ is the image of $\{u \in \mathcal{A}^* \mid uu^\circ \in Z(\mathcal{A})\}$ in $\overline{\mathcal{A}}$. Note that the index $N(\mathcal{V}) : N_z(\mathcal{V})$ is finite. As a point of general notation, $\mathrm{Inn}(\Gamma)$ will always denote the group of automorphisms of a ring $\Gamma$ induced by conjugating with units in $\Gamma$, and $\mathrm{Out}(\Gamma) := \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$.

PROPOSITION 4.5.

(i) *$N(L)$ acts on $\mathrm{Bil}_\Lambda(L)$ with kernel $U(L)$.*

(ii) *$N(L)$ acts on $\mathrm{End}_\Lambda(L)$ via conjugation also with kernel $U(L)$. In particular,* $\mathrm{Aut}(\mathrm{Bil}_\Lambda(L))$ *embeds into* $\mathrm{Aut}(\mathrm{End}_\Lambda(L))$.

(iii) *$N(L)$ acts on $\Lambda(L)$ by conjugation with kernel $\mathrm{End}_\Lambda(L)^*$. The induced automorphisms respect the involution $^\circ$.*

(iv) *Denote the kernel of the conjugation action of $N(L)$ on $Z(\mathrm{End}_\Lambda(L)) = Z(\Lambda(L))$ (or on $Z(\mathcal{A}) = Z\big(\mathrm{End}_\mathcal{A}(\mathcal{V})\big)$) by $N_z(L)$. Then $N_z(L)$ is a normal subgroup of finite index in $N(L)$ containing $\langle \mathrm{End}_\Lambda(L)^*, U(L)\rangle$, which is also of finite index.*

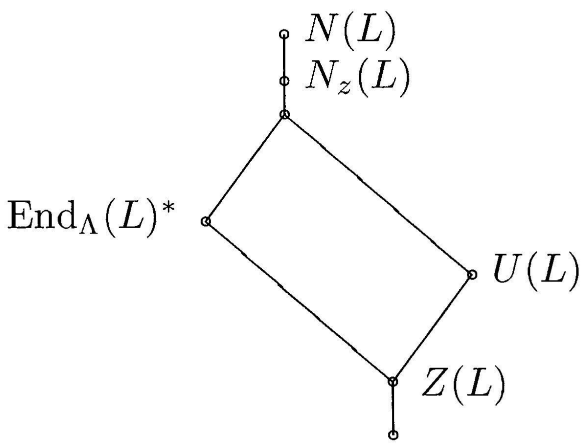
(v) *$\langle \mathrm{End}_\Lambda(L)^*, U(L)\rangle$ is a central product of $\mathrm{End}_\Lambda(L)$ and $U(L)$ amalgamated over $Z(L) := \mathrm{End}_\Lambda(L) \cap U(L)$.*

(vi) *The image of the conjugation action of $N_z(L)$ on $\Lambda(L)$ induces a finite index subgroup $\mathrm{Aut}_{z,L}(\Lambda(L), ^\circ)$ of $\mathrm{Aut}_z(\Lambda(L), ^\circ)$. The latter is also the image of the conjugation action of $\{u \in \widetilde{U}(\mathcal{A}, ^\circ) \mid u^{-1}\Lambda(L)u = \Lambda(L)\}$.*

(vii) *The image of the conjugation action of $N_z(L)$ on $\mathrm{End}_\Lambda(L)$ induces a subgroup $\mathrm{Aut}_{z,L}(\mathrm{End}_\Lambda(L))$ of $\mathrm{Aut}_z(\mathrm{End}_\Lambda(L))$. The latter is also the image of $\{\varphi \in \mathrm{End}_\mathcal{A}(\mathcal{V})^* \mid \varphi^{-1}\,\mathrm{End}_\Lambda(L)\varphi = \mathrm{End}_\Lambda(L)\}$*

(viii) *The group $N_z(L)/Z(L)$ is a subdirect product of $\mathrm{Aut}_{z,L}(\Lambda(L), ^\circ)$ and $\mathrm{Aut}_{z,L}(\mathrm{End}_\Lambda(L))$, amalgamated over the common finite factor group*

$$\mathrm{Aut}_{z,L}(\Lambda(L), ^\circ)/\mathrm{Inn}(\Lambda(L), ^\circ) \cong \mathrm{Aut}_{z,L}(\mathrm{End}_\Lambda(L))/\mathrm{Inn}(\mathrm{End}_\Lambda(L))$$
$$\cong N_z(L)/\langle \mathrm{End}_\Lambda(L)^*, U(L)\rangle\,.$$



*Proof:* Most of the statements can be verified in a straightforward way in the order in which they are listed, by using the preceding discussion of $N_z(\mathcal{V})$. The various finiteness statements follow from Proposition 3.5. □

Here is the main consequence for $\operatorname{Aut}(\operatorname{Bil}_\Lambda(L))$ :

COROLLARY 4.6. $\operatorname{Inn}(\operatorname{Bil}_\Lambda(L)) \trianglelefteq \operatorname{Aut}(\operatorname{Bil}_\Lambda(L))$, *and*

$$\operatorname{Out}(\operatorname{Bil}_\Lambda(L)) := \operatorname{Aut}(\operatorname{Bil}_\Lambda(L))/\operatorname{Inn}(\operatorname{Bil}_\Lambda(L))$$

*embeds into* $\operatorname{Out}(\operatorname{End}_\Lambda(L))$ *and into* $\operatorname{Out}(\Lambda(L),{}^\circ)$. *In particular,* $\operatorname{Out}(\operatorname{Bil}_\Lambda(L))$ *is finite.*

It is worthwhile to extract the following slightly more technical consequence as well.

COROLLARY 4.7. *Denote by* $\operatorname{Aut}_z(\operatorname{Bil}_\Lambda(L))$ *the group of automorphisms of* $\operatorname{Bil}_\Lambda(L)$ *induced by* $N_z(L)$. *Then*

$$\operatorname{Inn}(\operatorname{Bil}_\Lambda(L)) \trianglelefteq \operatorname{Aut}_z(\operatorname{Bil}_\Lambda(L)) \trianglelefteq \operatorname{Aut}(\operatorname{Bil}_\Lambda(L)),$$

$\operatorname{Aut}(\operatorname{Bil}_\Lambda(L))/\operatorname{Aut}_z(\operatorname{Bil}_\Lambda(L))$ *is isomorphic to a subgroup of the (obviously) finite group* $\operatorname{Aut}(Z(\Lambda(L))$, *and*

$$\operatorname{Out}_z(\operatorname{Bil}_\Lambda(L)) := \operatorname{Aut}_z(\operatorname{Bil}_\Lambda(L))/\operatorname{Inn}(\operatorname{Bil}_\Lambda(L))$$

*embeds into the finite groups* $\operatorname{Out}_z(\operatorname{End}_\Lambda(L))$ *and* $\operatorname{Out}_z(\Lambda(L),{}^\circ)$.

The next topics are the lattice versions of e-depth and e-$*$-depth, cf. Definition 3.6. Recall the notation introduced before Definition 3.6.

DEFINITION 4.8.
(i) Let $L^{(l)}$ be defined as $(\operatorname{End}_\Lambda(L))^{(l)}L$.
(ii) Define $L^{(0)} := L$ and $L^{(i)} := (L^{(i-1)})^{(l)}$, which yields an increasing sequence of full lattices in $\mathcal{V}$ :

$$L = L^{(0)} \leq L^{(1)} \leq L^{(2)} \leq \cdots$$

(iii) The length of this sequence, i. e. the first $i$ with $L^{(i)} = L^{(i+1)}$ is called the *depth* of $L$, resp. of $\operatorname{Bil}_\Lambda(L)$.

As a subtle point, note that $\operatorname{End}_\Lambda(L^{(1)})$ might contain $(\operatorname{End}_\Lambda(L))^{(1)}$ properly. In particular, $L$ is of depth 0 if and only if $\operatorname{End}_\Lambda(L)$ is hereditary, which is also equivalent to $\mathrm{Bil}_\Lambda(L)$ having e-depth zero. For these situations the Picard group techniques mentioned above can easily be applied. But before going into the details of the depth-zero case, a general remark on the smoothing process must be made.

REMARK 4.9. $N(L^{(i)})$ acts on $L^{(i+1)}$, i.e. $N(L^{(i)})$ is conjugate to a subgroup of $N(L^{(i+1)})$ under $\mathrm{GL}(\mathcal{V})$.

*Proof.* Clearly, the conjugation action by elements in $N(L^{(i)})$ preserves $\mathrm{arad}(\mathrm{End}_\Lambda(L^{(i)}))$ and therefore also the idealizer $(\mathrm{End}_\Lambda(L^{(i)}))^{(l)}$. But $L^{(i+1)} = (\mathrm{End}_\Lambda(L^{(i)}))^{(l)}L^{(i)}$. □

Continuing the discussion of e-$*$-depth of the last section, the notion of $*$-depth will be defined. Ideally one is tempted to imitate Definition 4.8 along the following lines: define $L^{(rl)}$ as the lattice in $\mathcal{V}$ containing $L$ with the property $\mathrm{End}_\Lambda(L \oplus L^*)^{(rl)}(L \oplus L^*) = L^{(rl)} \oplus M$ for some $\Lambda$-lattice $M$ in $\mathcal{V}$ containing $L^*$; define $L^{[0]} := L$ and $L^{[i]} := (L^{[i-1]})^{(rl)}$, which yields an increasing sequence of full lattices in $\mathcal{V}$:

$$L = L^{[0]} \leq L^{[1]} \leq L^{[2]} \leq \cdots ;$$

and define the $*$-depth of $L$, resp. $\mathrm{Bil}_\Lambda(L)$, to be the length of this sequence, i.e. the first $i$ with $L^{[i]} = L^{[i+1]}$.

To prove that everything is well defined, one needs a statement ensuring that this process really terminates. This boils down to: $\mathrm{End}_\Lambda\big(L^{(rl)} \oplus (L^{(rl)})^*\big)$ *contains* $\mathrm{End}_\Lambda(L \oplus L^*)$ *properly up to conjugation, unless* $\mathrm{End}_\Lambda(L \oplus L^*)$ *is hereditary.* I have not been able to prove this statement, though the argument below for the soundness of the less satisfactory definition, points somewhat in the right direction.

DEFINITION 4.10.

(i) Define sequences $L^{[0]} = L \leq L^{[1]} \leq L^{[2]} \ldots$ of lattices in $\mathcal{V}$ and $L^{*[0]} = L^* \leq L^{*[1]} \leq L^{*[2]} \ldots$ as follows:

$$L^{[i+1]} \oplus L^{*[i+1]} := \Gamma_i^{(rl)}(L^{[i]} \oplus L^{*[i]})$$

with $\Gamma_i := \mathrm{End}_\Lambda(L^{[i]} \oplus L^{*[i]}) \cap \mathrm{End}_\Lambda((L^{*[i]})^* \oplus (L^{[i]})^*)$.

(ii) Define the *$*$-depth* of $L$, resp. $\mathrm{Bil}_\Lambda(L)$, to be the length of these sequences, i.e. the first $i$ with $L^{[i]} = L^{[i+1]}$ and $L^{*[i]} = L^{*[i+1]}$.

Here is a verification that the definition makes sense.

LEMMA 4.11.

(i) $\Gamma_i(L^{[i]} \oplus (L^*)^{[i]})$ *decomposes as indicated in Definition 4.10 (i).*

(ii) *For the order* $\Gamma_i$ *one has* $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$, *so that the* $*$*-depth is well defined, namely as the first* $i$ *with* $\Gamma_i$ *hereditary.*

(iii) *Let* $s$ *be the* $*$*-depth of* $L$, *then* $\mathrm{End}_\Lambda(L^{[s]} \oplus (L^{[s]})^*)$ *is hereditary, i. e. the* $*$*-depth of* $L^{[s]}$ *is zero.*

*Proof.* (i) Since the two idempotents mapping $L \oplus L^*$ onto $L$, resp. $L^*$, lie in any of $\Gamma_i$, the result follows.

(ii) By definition $\Gamma_i^{(lr)} \subseteq \mathrm{End}_\Lambda(L^{[i+1]} \oplus L^{*[i+1]})$. Moreover $\Gamma_i^{(lr)}$ is invariant under the involution; by Remark 2.4, it is therefore also contained in $\mathrm{End}_\Lambda((L^{*[i+1]})^* \oplus (L^{[i+1]})^*)$. Hence $\Gamma_i \subseteq \Gamma_i^{(lr)} \subseteq \Gamma_{i+1}$.

(iii) $\Gamma_s$ is hereditary; hence $\mathrm{End}_{\Gamma_s}((L^{[s]} \oplus L^{*[s]}) \oplus ((L^{*[s]})^* \oplus (L^{[s]})^*)) \subset \mathcal{A}$ is hereditary. But $\Lambda(L^{[s]})$ contains this order and is therefore also hereditary, which makes $\mathrm{End}_\Lambda(L^{[s]} \oplus (L^{[s]})^*)$ hereditary. □

Various comments should be made. The notions of $*$-depth zero and e-$*$-depth zero are the same. This paper will mainly concentrate on the $*$-depth zero case, for which the two approaches yield the same answer. The first approach would in general be superior to the second one, because it defines a directed graph on the set of isomorphism classes of lattices in $\mathcal{V}$ with an arrow pointing from $L$ to $L^{[1]}$ (in the first meaning).

This would have the nice property that one has no cycles except for the one with $*$-depth $0$, and the depth of any lattice could be read off from the graph. In the second setting this is no longer possible. One has only an assignment to a $*$-depth zero lattice for any lattice without the intermediate steps. Example 2.2 (i) and Remark 2.8 show that one can produce situations where the $*$-depth is arbitrarily high with the depth being zero already.

It should be noted that this result implies a classical theorem by Watson, cf. [Wat62], which has been rediscovered by various people; and it puts the Watson process into the proper general framework. Strictly speaking, the assumption of positive definiteness is too strong, but it is retained here because it is the general hypothesis of the present paper. Various generalizations have been discarded, though they could have also been listed here.

COROLLARY (Watson). *Let* $L := \mathbf{Z}^{1\times n}$ *and* $\phi\colon L \times L \to \mathbf{Z}$ *be a (positive definite)* $\mathbf{Z}$*-bilinear form on* $L$*. Then there exists a full* $\mathbf{Z}$*-lattice* $M$ *in* $\mathcal{V} := \mathbf{Q} \otimes_{\mathbf{Z}} L$ *which is* $\mathrm{Aut}(L,\phi)$*-invariant and satisfies* $M \subseteq M^{\#} \subseteq k^{-1}M$ *for some square-free divisor* $k$ *of* $\det(L,\phi)$*, where the reciprocal lattice* $M^{\#}$ *is taken with respect to some rational multiple of* $\phi$*.*

*Proof.* $\phi$ induces an involution on $\mathcal{A} := \mathbf{Q}^{n\times n}$ containing $\Lambda := \Lambda(L)$ as an invariant $\mathbf{Z}$-order. Denote the $*$-depth of $L$ by $s$ and set $M := L^{[s]}$. Clearly, $\mathrm{Aut}(L,\phi) = U(L)$, and $M$ is $U(L)$-invariant. Since $\Gamma := \mathrm{End}_{\Lambda}(M \oplus M^*)$ is hereditary, the same applies to $\Lambda(M)$ $(= \mathrm{End}_{\Gamma}(M \oplus M^*))$. By the general properties of hereditary orders (as chain orders), the claim follows, since $M^{\#}$ is an absolutely irreducible $\Lambda$-lattice isomorphic to $M^*$. □

Having a canonical procedure for constructing $*$-depth zero lattices from ones of arbitrary $*$-depth such that the statements of Remark 4.9 carry over, it becomes an interesting question to look into the structure of $\mathrm{Out}_z(\mathrm{Bil}_{\Lambda}(L))$ in this case. Of course, it is no loss of generality if one restricts to the case of simple algebras $\mathcal{A}$. Here is a first statement, whose hypothesis is often satisfied.

THEOREM 4.12. *Let* $L$ *be of* $*$*-depth zero and assume that the centre* $Z(\mathcal{A})$ *is a totally real number field. Then* $\mathrm{Out}_z(\mathrm{Bil}_{\Lambda}(L))$ *is of exponent dividing* 2.

*Proof.* Because of Proposition 4.5 (vi) and Corollary 4.7, one has to prove the following: for $u \in \widetilde{U}(\mathcal{A}) \cap N(\Lambda(L))$ the square $u^2$ induces an inner automorphism of $\Lambda(L)$. Let $uu^{\circ} = z$ for some element $z \in Z(\mathcal{A})$. Then $u^2$ and $z^{-1}u^2$ induce the same automorphism. But $z^{-1}u^2$ lies is $U(\mathcal{A})$, since $z^{\circ} = z$. Each prime of $Z(\mathcal{A})$ is mapped onto itself by the involution $^{\circ}$. Hence, at the completion of the whole situation at any prime $\mathfrak{p}$ of $Z(\mathcal{A})$, the element $z^{-1}u^2$ again lies in a unitary group and cannot induce a shift on the irreducible lattices in the sense of the proof of Lemma 3.8. It therefore lies in any completion of $\Lambda(L)$ and hence in $\Lambda(L)$. Since $\Lambda(L)$ is invariant under the involution, also the inverse of $z^{-1}u^2$ lies in $\Lambda(L)$ and the claim follows. □

Here is a $*$-depth zero example, where the hypothesis of Theorem 4.12 is violated and $\mathrm{Out}_z(\mathrm{Bil}_{\Lambda}(L))$ is of order 3.

EXAMPLE 4.13. Let $G := \langle a,b,c \mid a^7, b^3, a^b = b^2, c^3, [a,c], [b,c] \rangle$ be the group $(C_7 : C_3) \times C_3$ and let $\Lambda$ be the residue class order of $\mathbf{Z}G$ modulo the ideal generated by $a-1$ and $c-1$. Then $\mathcal{A} \cong K^{3\times 3}$ with $K = \mathbf{Q}[\sqrt{-3}, \sqrt{-7}]$ (of class number 1, cf. [PoZ89]) and

$$\Lambda \cong \begin{pmatrix} R & R & R \\ I & R & R \\ I & I & R \end{pmatrix},$$

where $R = \mathbf{Z}[\frac{-1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-7}}{2}] = \mathbf{Z}_K$ is the maximal $\mathbf{Z}$-order in $K$ and $I$ is the product of the two prime ideals $I_1$ and $I_2$ above 7 in $K$, i.e. $7R = I^2$. The natural involution of $\mathbf{Q}G$ induces the involution $^\circ$ of $\mathcal{A}$ of interest. Finally, $L := \Lambda_\Lambda$ is chosen as the regular $\Lambda$-lattice, i.e. with respect to the above description of $\Lambda$, one has $L = (R,R,R) \oplus (I,R,R) \oplus (I,I,R)$. One easily checks that the group automorphism $a \mapsto a$, $b \mapsto bc$, $c \mapsto c$ maps $\Lambda$ onto itself and things can be arranged so that $(R,R,R)$ is mapped onto $(I_1, R, I_2^{-1})$, and $(I_1, R, I_2^{-1})$ onto $(I_1, I_1 I_2^{-1}, I_2^{-1})$. Since, clearly, $L \cong (R,R,R) \oplus (I_1, R, I_2^{-1}) \oplus (I_1, I_1 I_2^{-1}, I_2^{-1})$, this reveals an element of order 3 in $\mathrm{Aut}_z(\mathrm{Bil}_\Lambda(L))/\mathrm{Inn}(\mathrm{Bil}_\Lambda(L))$. In fact, $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ is of order 12.

The general situation for the $*$-depth zero case is as follows with the notation of Definition 3.10.

THEOREM 4.14. *Let $L$ be of $*$-depth zero and assume (w.l.o.g.) that $\mathcal{A}$ is simple. Then* $\mathrm{Out}_z(\mathrm{Bil}_\Lambda(L))$ *is Abelian and embeds into an extension of the class group $Cl(Z(\Lambda(L)))$ of the centre $Z(\Lambda)$ by a group of the form*

$$\bigoplus_{\mathfrak{p}\in\mathcal{S}} C_2 \oplus \bigoplus_{\{\mathfrak{p},\mathfrak{p}^\circ\}\in\mathcal{N}} C_{s(\Lambda(L),\mathfrak{p})},$$

*with $\mathcal{S}$ the set of prime ideals $\mathfrak{p}$ of $Z(\mathcal{A})$ with $\mathfrak{p} = \mathfrak{p}^\circ$ and $s(\Lambda(L),\mathfrak{p})$ even, and $\mathcal{N}$ the set of pairs $\{\mathfrak{p},\mathfrak{p}^\circ\}$ of prime ideals with $\mathfrak{p} \neq \mathfrak{p}^\circ$.*

*Proof.* That $\mathrm{Out}(\mathrm{Bil}_\Lambda(L))$ is Abelian was already shown in Theorem 3.9. As in the proof of Theorem 4.12, let $u \in \widetilde{U}(\mathcal{A}) \cap N(\Lambda(L))$. At each prime $\mathfrak{p}$ of $Z(\mathcal{A})$, $u$ induces a shift of period $a(\mathfrak{p}) \mid s(\Lambda(L),\mathfrak{p})$, as explained in Lemma 3.8 and Definition 3.10. Let $uu^\circ = z$ for some element $z \in Z(\mathcal{A})$. At the real primes $\mathfrak{p} = \mathfrak{p}^\circ$, both $u$ and $u^\circ$ shift by the same index, and hence the induced shift generates at most a subgroup of order 2 of $C_{s(\Gamma(L),\mathfrak{p})}$. If $\mathfrak{p} \neq \mathfrak{p}^\circ$, the induced shifts at $\mathfrak{p}$ and $\mathfrak{p}^\circ$ are opposite to each other and of the same order modulo local central shifts. Since the situation is global, the class group of the centre has to be taken into account, as in the proof of Lemma 3.8. □