Zeitschrift: L'Enseignement Mathématique
Band: 47 (2001)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LATTICES OF COVARIANT QUADRATIC FORMS
Kapitel: 6. Some three-dimensional lattices of covariant forms
Autor: Plesken, Wilhelm
DOI: https://doi.org/10.5169/seals-65427

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EXAMPLE 5.8.

(i) Of the four irreducible Bravais groups of degree 8 whose commuting algebra is a nonsplit rational quaternion algebra (ramified at 2 and 3), cf. [Sou94], the e of Proposition 5.7 is 1, 2, 3 and 6. In all cases $\mathrm{Bil}_\Lambda^+(L)$ is modular and c_0 is equal to 1. In [Neb99] the Hermite function on the fundamental domains for these cases is plotted.

(ii) In Example 2.2 (ii), choose f_0 to be m-modular for some natural number m. Then $\mathrm{Bil}_\Lambda^+(L \oplus L)$ (in the notation of Example 2.2 (ii)) is modular, where the e of Proposition 5.7 is equal to m, as is c_0.

To test whether $\mathrm{Bil}_\Lambda^+(L)$ is modular, one can simply compute the images of a $\mathbf{Z}$-basis of $\mathrm{Bil}_\Lambda^+(L)$ under ι as described in Theorem 5.5 and find a simultaneous isometry of L to L^* (with respect to all of the forms, resp. their images). For this there is a powerful algorithm with implementation available, cf. [PlS97]. Instead of a whole basis, it is sometimes enough to look at one sufficiently general form; details on this will be given in a subsequent paper, as well as some examples with $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2\times 2}$. One such example, involving the Leech lattice with $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ a non-split quaternion algebra over $\mathbf{Q}[\sqrt{-7}]$, is sketched in the last chapter of [Ple96].

6. SOME THREE-DIMENSIONAL LATTICES OF COVARIANT FORMS

This chapter is devoted to some examples in the case where $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$ and where the depth of $\mathrm{Bil}_\Lambda(L)$ is 0. The typical questions we try to answer are: how to relate the various invariants? are outer automorphisms possible? are modular lattices possible? how does the automorphism group of $\mathrm{Bil}_\Lambda^+(L)$ compare to the orthogonal group of $(\mathrm{Bil}_\Lambda^+(L), q)$? The simplest case is $\mathrm{End}_\Lambda(L) \cong \mathbf{Z}^{2\times 2}$, where all these questions can be answered.

THEOREM 6.1. *Let* $\mathrm{End}_\Lambda(L) \cong \mathbf{Z}^{2\times 2}$. *Then* $L = L_0 \oplus L_0$ *for some irreducible* Λ*-lattice* L_0. *Let* ϕ_0 *be the positive definite generator of* $\mathrm{Bil}_\Lambda^+(L_0)$. *Then* c, c_0, *and* q, *introduced in Theorem 5.5, are as follows.*

(i) *With respect to a suitable basis of* $\mathrm{Bil}_\Lambda^+(L)$, *the quadratic form* q *of Theorem 5.5 becomes* $xy - z^2$.

(ii) $c = \det(\phi_0)^2$.

(iii) c_0 *is the exponent of* $L_0^\sharp/L_0$, *i.e. the biggest elementary divisor of a Gram matrix of* ϕ_0.

(iv) $\mathrm{Inn}(\mathrm{Bil}_\Lambda^+(L)) = \mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$.

(v) $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ *is of index* 2 *in* $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$. *More precisely, it is equal to the kernel of* $-\theta$ *intersected with* $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L))$, *where* θ *is the spinor norm of* $\mathrm{O}(\mathrm{Bil}_\mathcal{A}^+(\mathcal{V}), q)$ *([Scha85], p. 336).*

(vi) *The nondegenerate* $\phi \in \mathrm{Bil}_\Lambda^+(L)$ *are modular if and only if* ϕ_0 *is* c_0*-modular. In this case such a* ϕ *is* $c_0 q(\phi)$*-modular.*

(vii) *The e-∗-depth of* $\mathrm{Bil}_\Lambda(L)$ *is given by* $[\frac{r}{2}]$, *where* r *is maximal with* $p^r \mid c_0$ *for some prime number* p.

Proof. Choose a basis for L_0. This yields a Gram matrix A of ϕ_0. With respect to a suitable basis of L, one gets $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right)$ as Gram matrices for the obvious basis of $\mathrm{Bil}_\Lambda^+(L)$. Since $\det\left(\left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right) \otimes A\right) = \det(A)^2(xy - z^2)^m$ and $\left(\left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right) \otimes A\right)^{-1} = (xy - z^2)^{-1}\left(\begin{smallmatrix} y & -z \\ -z & x \end{smallmatrix}\right) \otimes A^{-1}$, the claims (i) to (iv) follow. (v) is straightforward with [Mac81]. (vi) and (vii) are obvious. □

The general case of depth 0 is more involved:

PROPOSITION 6.2. *Assume* $\mathcal{E} \cong \mathbf{Q}^{2\times 2}$ *and* L, *resp.* $\mathrm{Bil}_\Lambda^+(L)$, *is of depth* 0. *Let* $d := p_1 \cdots p_k$ *be the product of the different primes at which* $\mathrm{End}_\Lambda(L)$ *is not maximal.*

(i) *There are unique natural numbers* s, t *such that the quadratic form* q *on* $\mathrm{Bil}_\Lambda^+(L)$ *described in Theorem 5.5 becomes* $sxy - tz^2$ *with respect to any basis* (ϕ, ψ, χ) *of* $\mathrm{Bil}_\Lambda^+(L)$ *such that* $\phi, \psi \in \mathrm{Bil}_{\Lambda,\geq 0}^+(L)$ *with* $L = Rad_\psi(L) \oplus Rad_\phi(L)$ *and* χ *is zero on both direct summands. The product* st *divides* d.

(ii) *The constant* c *of Theorem 5.5 is given by*

$$c = \det(\overline{\phi})\det(\overline{\psi})s^{-m},$$

where $2m = \dim_{\mathbf{Q}}(\mathcal{V})$, $\overline{\phi}$ *is the scalar product on* $Rad_\psi(L)$ *induced by* ϕ, *and* $\overline{\psi}$ *the scalar product on* $Rad_\phi(L)$ *induced by* ψ.

Note that, providing $k > 0$, there are 2^{k-1} such bases up to interchanging ϕ and ψ and up to $\mathrm{End}_\Lambda(L)$ operation.

Proof. Let $L = L_1 \oplus L_2$ with absolutely irreducible Λ-lattices L_1, L_2. One may assume $dL_1 \leq L_2 \leq L_1$. Note this implies that L_1^* can be considered to sit inside L_2^* with $L_1^* \leq L_2^* \leq d^{-1}L_1^*$. As a result, $\mathrm{Hom}_\Lambda(L_1, L_2^*) = d_1 \mathrm{Hom}_\Lambda(L_1, L_1^*)$ for some divisor d_1 of d, and

$\mathrm{Hom}_\Lambda(L_2, L_2^*) = d_2 \,\mathrm{Hom}_\Lambda(L_1, L_2^*)$ for some divisor d_2 of d. Introducing a basis for L_1, as with L_0 in Theorem 6.1, identifies $L_1 =: L_0$ with $\mathbf{Z}^{1\times m}$; and choosing a basis for L_2 identifies L_2 with $\mathbf{Z}^{1\times m}T$, where $T \in \mathbf{Z}^{m\times m}$ represents the change of bases. Denote the $m\times m$-unit matrix by $I = I_m$. The computation for Theorem 6.1 can be transformed as follows:

$$\begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} x'A & z'A \\ z'A & y'A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix}^{tr} = \begin{pmatrix} xA & zd_1^{-1}AT^{tr} \\ zd_1^{-1}TA & yd_1^{-1}d_2^{-1}TAT^{tr} \end{pmatrix},$$

with $x = x', z = d_1 z', y = d_1 d_2 y'$. The parameter choice $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ for (x,y,z) yields a typical basis for $\mathrm{Bil}_\Lambda^+(L)$ as described above. Taking determinants yields

$$\det(T)^2 \det(A)^2 \Big(\frac{xy}{d_1 d_2} - (\frac{z}{d_1})^2\Big)^m,$$

and hence (i) and (ii) with $s = d_1 g^{-1}, t = d_2 g^{-1}$ relatively prime, where $g = \gcd(d_1, d_2)$, if one uses $\det(\overline{\phi}) = \det(A)$. That s,t do not depend on the particular decomposition of L follows from analyzing the determinant of q. □

Working through the various cases for determining c_0 in Theorem 5.5 is left as an exercise. Before analyzing $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ one needs to look at the automorphism groups of the quadratic forms involved. Note that the automorphism groups of $kxy - z^2$ for $k \in \mathbf{N}$ square free are analyzed in quite some detail in [Mac81]. In the present context two extra details are needed.

LEMMA 6.3. *Let $s,t \in \mathbf{N}$ be square free and relatively prime, and let $k := st$.*

(i) *The diagonal matrix* $\mathrm{diag}(t,t,1)$ *transforms* $\mathrm{O}(\mathbf{Z}^{1\times 3}, sxy - tz^2)$ *onto* $\mathrm{O}(\mathbf{Z}^{1\times 3}, kxy - z^2)$.

(ii) *There is an exact sequence of groups:*

$$\langle -I_2 \rangle \hookrightarrow \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z} \end{pmatrix}^* \to \mathrm{O}(\mathbf{Z}^{1\times 3}, kxy - z^2) \to D_k \to 1,$$

where $D_k \le \mathbf{Q}^*/(\mathbf{Q}^*)^2$ is generated by the cosets of the divisors d of k (including -1).

Proof. (i) Denote the quadratic forms $sxy - tz^2$ and $kxy - z^2$ by q and q' respectively. On $L = \mathbf{Z}^{1\times 3}$ they define integral bilinear forms b and b', e.g. $b(l_1, l_2) = q(l_1 + l_2) - q(l_1) - q(l_2)$ for $l_1, l_2 \in L$. Clearly, $\mathrm{O}(L,q)$ also acts on the reciprocal lattice $L^\sharp$ of L with respect to b, and $\mathrm{O}(L,q')$ also acts on the reciprocal lattice $L'^{\sharp}$ of L with respect to b'. Hence $\mathrm{diag}(t,t,1)$, which maps L onto $tL^{\sharp}\cap L$ and q onto tq', conjugates $\mathrm{O}(L,q)$ into $\mathrm{O}(L,q')$. For the reverse inclusion one argues similarly for t odd with $tL'^{\sharp}\cap L$ and one has to work with $\frac{t}{2}L'^{\sharp}\cap L$, taking the even sublattice, for t even.

(ii) Define $L_d := \{\left(\begin{smallmatrix} a & c \\ c & db\end{smallmatrix}\right) \mid a,b,c\in \mathbf{Z}\}$ and consider the determinant det as a quadratic form on L_d for any natural number d. Then $(L_k, \det)$ is isometric to $(\mathbf{Z}^{1\times 3}, kxy - z^2)$. One easily checks that $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$ acts on L_k by $X\mapsto gXg^{tr}$ for all $X\in L_k$ and $g\in \left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$. Clearly this action respects the determinant, whence the exactness of the left half of the sequence is established. Note, for $k=1$, the full claim was already proved in Theorem 6.1. Clearly $L_k\le L_1$ and the stabilizer S_k of L_k in $\mathrm{O}(L_1,\det)$ is generated by $-id_{L_1}$ and the image of $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$. As in Theorem 6.1 denote the spinor norm of $\mathrm{O}(\mathbf{Q}^{1\times 3}, xy-z^2)$ by θ. Then $-\theta$ restricted to $\mathrm{O}(L_k,\det)$ will be the homomorphism on the right of the exact sequence. Clearly the image of $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$ is in the kernel of $-\theta$. To complete the proof, it is enough to show, by induction on the number $d(k)$ of prime divisors of k, that $\mathrm{O}(L_k,\det)$ contains S_k of index $2^{d(k)}$ and is generated by an S_k and elements (Atkin-Lehner involutions) mapped by $-\theta$ onto $p(\mathbf{Q}^*)^2$ for the primes p dividing k.

The statement follows for $d(k)=1$, i.e. $k=p$ prime, as follows: the orbit of L_1 under $\mathrm{O}(L_p,\det)$ consists of L_1 and $L_{1,p}$, where in general $L_{1,d} := \{\left(\begin{smallmatrix} d^{-1}a & c \\ c & db\end{smallmatrix}\right) \mid a,b,c\in\mathbf{Z}\}$. This is because L_1 must be mapped onto an isometric lattice contained in $L_p^{\sharp}$ and containing L_p. The isometry fixing L_p and mapping L_1 onto $L_{1,p}$ is the reflection by the vector $\mathrm{diag}(-1,p)\in L_p$, which can also be realized by extending the operation via 2×2-matrices to $p^{-\frac{1}{2}}\left(\begin{smallmatrix} 0 & 1 \\ p & 0\end{smallmatrix}\right)$. This settles the case $d(k)=1$. Now assume the statement proved for $\mathrm{O}(L_d,\det)$ for all proper divisors d of k. Let $k=pk'$ for some prime divisor p of k. Obviously the orbit of $L_{k'}$ under the action of $\mathrm{O}(L_{k'},\det)$ is of length $p+1$, as is the orbit under $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Z} \\ k'\mathbf{Z} & \mathbf{Z}\end{smallmatrix}\right)^*$. Hence, the stabilizer of L_k in $\mathrm{O}(L_{k'},\det)$ is an extension of S_k by an elementary Abelian 2-group of rank $d(k)-1=d(k')$. An argument similar to the one above shows that this stabilizer is of index at most 2 in $\mathrm{O}(L_k,\det)$. That it is of index exactly 2 can then be seen via the element of $\mathrm{O}(L_p,\det)$ with spinor norm $-p$. (In [Que96] the precise element is given, cf. also [Mac81].)

Note, the elementary Abelian 2-group $\mathrm{O}(L_k,\det)/S_k$ acts regularly on the set $\{L_{1,d} \mid d \text{ divides } k\}$. In terms of the affine building belonging to the p-adic completion of the group, all $L_{1,d}$ with $p\nmid d\mid k$ belong to one vertex of the attached tree and all other $L_{1,d}$ belong to a different vertex, which is not of the same type as the first vertex. Finally L_p, resp. all L_d with $p\mid d$, belong to the edge connecting the two vertices. □

Now Proposition 6.2 can be completed:

PROPOSITION 6.4. *Under the hypothesis and notation of Proposition 6.2 the index of* $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ *in* $O(\mathrm{Bil}_\Lambda^+(L), q)$ *is* $2^{1+a}\prod(p+1)$, *where* p *runs through all prime divisors of* $\frac{d}{st}$ *and* a *is at most equal to the number of prime divisors of* st. *Moreover,* $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))/\mathrm{Inn}(\mathrm{Bil}_\Lambda^+(L))$ *is an elementary* 2*-group of rank* a.

Proof. This is an immediate consequence of Proposition 6.2 and Lemma 6.3. □

The question arises, whether there are examples for which the minimal possible index of $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ in $O(\mathrm{Bil}_\Lambda^+(L), q)$ according to Proposition 6.4 is attained, i.e. $a = 0$ and $d = st$. This is already possible in the group case; cf. Example 2.2 (ii).

PROPOSITION 6.5. *For a prime number* p *let* $c(p) = p - 1$ *if* p *is odd and* $c(2) = 2$. *Then, for any sequence of prime numbers* $p_1 < p_2 < \cdots < p_l$, *there are examples with* $\dim_{\mathbf{Q}} \mathcal{V} = 2\prod_{i=1}^{l} c(p_i)$, *where* $\mathcal{A}$ *is an image of a finite group algebra and* $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$, *where* $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ *is of (minimal) index* 2 *in* $O(\mathrm{Bil}_\Lambda^+(L), q)$. *If* $p_i \equiv 3 \pmod 4$ *for all* i *with* $p_i \neq 2$, *then* L *can be chosen so that each* $\phi \in \mathrm{Bil}_\Lambda^+(L)$ *is* $c_0 q(\phi)$*-modular.*

Proof. First construct a finite $\mathbf{C}$-irreducible subgroup $G(p)$ of $\mathrm{GL}_{c(p)}(\mathbf{Q})$ as follows: for $p = 2$ take the automorphism group of the quadratic lattice (which is a dihedral group of order 8); for p odd take the Frobenius group of order $p(p-1)$ in its action on the permutation module factored by the fixed points, which is then identified with $\mathbf{Q}^{1\times c(p)}$. Take the span of $-I_{c(p)}$ with this group to obtain $G(p) \leq \mathrm{GL}_{c(p)}(\mathbf{Q})$ of order $2p(p-1)$. The $G(p)$-lattices in $\mathbf{Q}^{1\times c(p)}$ are described in [NeP95a] p. 29: up to multiples they come in a chain $L_0(p) \geq L_1(p) \geq \cdots \geq L_{c(p)} = pL_0(p) \geq \cdots$, where $L_i(p)$ is of index p^i in $L_0(p)$. There exists an element n in the normalizer of $G(p)$ in $\mathrm{GL}_{c(p)}(\mathbf{Q})$ mapping $L_i(p)$ onto $L_{i+c(p)/2}(p)$. Choosing $L = L_i(p) \oplus L_{i+c(p)/2}(p)$ and taking the $G(p)$-invariant symmetric bilinear forms for $\mathrm{Bil}_\Lambda^+(L)$ gives the desired result for the case $d = p$. The case $i = 0$ for $p = 2$, resp. $i = \frac{p-3}{4}$ for $p \equiv 3 \pmod 4$, gives modularity. The general case of composite d is obtained from the above by taking tensor products. □

One should note that in the above proof one gets modular lattices by choosing $L = L_i \oplus L_{p-i}$ without having the big $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L)$, if i is not chosen as above. The same holds for the composite case. By now it should be clear that the existence of outer automorphisms and modularity of the lattices are different phenomena.

To end up, some explicit examples of $*$-depth zero will be given, where $\mathrm{End}_\Lambda(L) \cong \left(\begin{smallmatrix} \mathbf{Z} & 3\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{smallmatrix}\right)$. One easily checks that the unit group is generated by a, b, c and that the outer automorphism is induced by d with

$$a := \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d := \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}.$$

Note that defining relations for the inner, resp. outer, automorphism group are provided by $\bar{a}^2$, $\bar{b}^2$, $\bar{c}^2$, $(\bar{a}\bar{b})^3$ and $\bar{b}^2$, $\bar{c}^2$, $\bar{d}^2$, $(\bar{c}\bar{d})^2$, $(\bar{d}\bar{b})^6$ respectively. The fundamental domains in the hyperbolic plane identified with $\mathrm{Bil}^+_{\Lambda_\mathbf{R},>0}(\mathcal{V})/\mathbf{R}_{>0}$, where $\mathbf{R}_{>0}$ acts by multiplication, are triangles with vertices P, C_1, C_2 in the first case, where C_1 and C_2 are cusps, and P, C_1, M in the second case. The angles can be read off from the presentation. According to Example 3.7 there are seven possibilities for the equivalence type of $\mathrm{Bil}_\Lambda(L)$, parametrized by the exponent matrices of $\mathrm{End}_\Lambda(L \oplus L^*)$ given there. Only in four cases can one have outer automorphisms.

EXAMPLE 6.6.

(i) Take the fourth possibility in the list of Example 3.7. Then $L = L_1 \oplus L_2$ with $L_1^\# = L_1$ and $L_2^\# = 3L_2$, where the reciprocal lattices are taken with respect to a generator ϕ_1 of $\mathrm{Bil}_\Lambda(L_1)$, and $L_2 \leq L_1$ is necessarily of index $3^{n/2}$ in L_1 with $n := \dim(L_1)$. (Note: n must be even.) Representing $\mathrm{Bil}_\Lambda^+(L)$ by Gram matrices one gets $\mathrm{Bil}_\Lambda^+(L) = \{\left(\begin{smallmatrix} \alpha F_1 & \gamma X \\ \gamma X^{tr} & \beta F_2 \end{smallmatrix}\right) \mid \alpha, \beta, \gamma \in \mathbf{Z}\}$, where F_1 and F_2 are unimodular (Gram matrices for L_1 and L_2) and $XF_1^{-1}X^{tr} = 3F_2$. Obviously one has no outer isomorphism if F_1 and F_2 are not equivalent. In this case $\mathrm{Bil}_\Lambda^+(L)$ is not modular, though ι is bijective, but it is not an equivalence. In any case, the vertices of the fundamental domain in this case are given by the $(\alpha, \beta, \gamma) \in \{(2,2,1), (1,0,0), (0,0,1)\}$ corresponding to P, C_1, C_2, the determinant is $(\alpha\beta - 3\gamma^2)^n$ and a nice realization of this setup is for $n = 12$, where one can find the 3-scaled version of the unimodular lattice D_{12}^+ as a sublattice of the standard lattice of index 3^6. Things can be so chosen that the 2-fold cover of the Mathieu group M_{12} acts. In $\mathrm{Bil}^+_{\Lambda,>0}(L)$ one has two orbits of primitive M_{12}-perfect lattices, one unimodular with minimum 2 and one of determinant 5^{12} with minimum 4. Obviously one can produce many more examples in higher dimensions. One can show that there is no realization of this situation for $n < 12$.

If one has an outer automorphism, there seems to be the possibility that $\mathrm{Bil}_\Lambda^+(L)$ is modular. The vertices of the fundamental domain in this case are given by the $(\alpha,\beta,\gamma) \in \{(2,2,1), (1,0,0), (1,1,0)\}$ corresponding to P, C_1, M. For the case $F_1 = F_2$ I have computed some examples: $F_1 = I_4$, E_8, Λ_{24} (Leech lattice). In the first case the vertex P represents the root lattice E_8, which is the only perfect lattice here. In the other two cases my choice of X (there might be more than one!) yielded a 6-modular form as the only perfect form; the coordinates were $(3,3,1)$, the minima were 6 and 12 respectively.

(ii) Take the eighth possibility in the list of Example 3.7. Then $L = L_1 \oplus L_2$ with $3L_1 < L_2 = 3L_1^\# < L_1 = 3L_2$, where the reciprocal lattices are taken with respect to a generator ϕ_1 of $\mathrm{Bil}_\Lambda(L_1)$.

Again representing $\mathrm{Bil}_\Lambda^+(L)$ by Gram matrices with respect to suitably chosen bases one gets $\mathrm{Bil}_\Lambda^+(L) = \{\left(\begin{smallmatrix} \alpha F & \gamma I_n \\ \gamma I_n & \beta \tilde{F} \end{smallmatrix}\right) \mid \alpha,\beta,\gamma \in \mathbf{Z}\}$, where F are the Gram matrices for (L_1,ϕ_1) and $\widetilde{F} = 3F^{-1}$. The determinant is $(3\alpha\beta - \gamma^2)^n$. Obviously one has an outer isomorphism if and only if F and $\widetilde{F}$ are $\mathbf{Z}$-equivalent, i.e. if (L_1,ϕ_1) is 3-modular. Many such examples, with and without outer automorphisms and also for other exponents different from 3 of $L_1^\#/L_1$, have been investigated in [Bav97], because in this case $\mathrm{Bil}_\Lambda^-(L)$ is spanned by unimodular symplectic forms. By Proposition 5.7 $\mathrm{Bil}_\Lambda^+(L)$ is modular. Here are some examples with outer automorphisms: $F = A_2$, $A_2 \otimes E_8$, K_{12} (the Coxeter-Todd lattice), and $[\pm S_6(3) \overset{2}{\square} C_3]_{26}$ of [Neb96b]; one gets one relative extremal lattice with coordinates $(\alpha,\beta,\gamma) = (1,1,1)$. They are 2-modular with minima 2, 4, 4, and 6 respectively. However, $F = [\mathrm{SL}_2(9) \underset{\infty,3}{\overset{2(3)}{\otimes}} \mathrm{SL}_2(9).2]_{16}$, which is also 3-modular with minimum 4 of dimension 16 (like $A_2 \otimes E_8$), yields the 11-modular form with minimum 12 and coordinates $(\alpha,\beta,\gamma) = (3,3,4)$ as extremal lattice. Finally, $F = N_{23}$ (the extremal 3-modular lattice of dimension 24 of [Neb95]; or [Neb98b], Theorem 5.1 for an alternative construction) yields a 23-modular lattice as extremal with minimum $24 = 4 \cdot 6$ and coordinates $(\alpha,\beta,\gamma) = (4,4,5)$. It would be interesting to investigate the density function on the fundamental domain theoretically.