2.2 Symplectic characteristic classes and Chern classes

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.07.2024**

$\phi(\widetilde{\rho}(z))$ of any generator $z \in \mathbf{Z}/p\mathbf{Z}$ is conjugate to a $Y \in \mathrm{Sp}(2n, \mathbf{Z})$. Then $d_j(\rho) = \widetilde{\rho}^*(c_j) = c_j(\rho)$. We define the total Chern class of a representation $\widetilde{\rho}$ to be

$$c(\widetilde{\rho}) := 1 + c_1(\widetilde{\rho}) + c_2(\widetilde{\rho}) + \cdots + c_n(\widetilde{\rho}).$$

It has the well-known properties $c(\rho \oplus \sigma) = c(\rho)c(\sigma)$, $c(m\rho) = c(\rho)^m$, where ρ, σ are representations and m is a positive integer.

2.2 Symplectic characteristic classes and Chern classes

THEOREM 2.1. *Let p be an odd prime. Then for any $n = 1, \ldots, (p-1)/2$ there exists a representation $\widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}((p-1)/2)$ such that the n-th Chern class $c_n(\widetilde{\rho})$ is nonzero and the representation $\phi \circ \widetilde{\rho}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\rho\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$.*

The representation $\widetilde{\rho}$ factors through $\mathrm{Sp}(p-1, \mathbf{Z})$ if the image $\widetilde{\rho}(z)$ of a generator $z \in \mathbf{Z}/p\mathbf{Z}$ satisfies the condition stated in Theorem 1.2. Then, because $c_n(\widetilde{\rho}) \neq 0$, we have $d_n(\rho) \neq 0$ where $\rho\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$ is the representation corresponding to $\widetilde{\rho}$.

Proof of Theorem 2.1. Let $\mathcal{U}$ be the set of subsets $\mathcal{I} \subset (\mathbf{Z}/p\mathbf{Z})^*$ of cardinality $|\mathcal{I}| = (p-1)/2$, and $j \in \mathcal{I}$ implies $p - j \notin \mathcal{I}$. The cardinality of $\mathcal{U}$ is $2^{(p-1)/2}$. We always assume the elements $j \in \mathcal{I}$ to be represented by integers j with $1 \leqslant j < p$. Note that we will use the same notation for the elements of $\mathcal{I}$ and their representatives. For $j = 1, \ldots, p-1$ let $\widetilde{\rho}_j\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}(1)$ be the one-dimensional representation with $\widetilde{\rho}_j(z) := e^{j2\pi i/p}$ for a fixed generator $z \in \mathbf{Z}/p\mathbf{Z}$. For a given $\mathcal{I}$ we define $\widetilde{\rho}_{\mathcal{I}}$ to be the direct sum of the representations $\widetilde{\rho}_j$, $j \in \mathcal{I}$. Let $x := c_1(\widetilde{\rho}_1)$, then the total Chern class of $\widetilde{\rho}_{\mathcal{I}}$ is

$$c(\widetilde{\rho}_{\mathcal{I}}) = c\Big(\bigoplus_{j \in \mathcal{I}} \widetilde{\rho}_j\Big) = \prod_{j \in \mathcal{I}} (1 + jx).$$

The representations $\widetilde{\rho}_{\mathcal{I}}$ are those which factor through $\mathrm{Sp}(p-1, \mathbf{Z})$. For a given $\mathcal{I} \in \mathcal{U}$ we define $-\mathcal{I} := \{p - j \mid j \in \mathcal{I}\}$. Then $-\mathcal{I} \in \mathcal{U}$ and $\mathcal{I} \cup -\mathcal{I} = (\mathbf{Z}/p\mathbf{Z})^*$. Moreover, we get $c(\widetilde{\rho}_{\mathcal{I}})c(\widetilde{\rho}_{-\mathcal{I}}) = 1 - x^{p-1}$. The n-th Chern class $c_n(\widetilde{\rho}_{\mathcal{I}})$ is nonzero if and only if the coefficient a_n of x^n in the total Chern class $c(\widetilde{\rho}_{\mathcal{I}})$ is nonzero. Let $\mathcal{I} := \{j_1, \ldots, j_{(p-1)/2}\} \in \mathcal{U}$; then we define

$$\mathcal{I}_l := \{j_1, \ldots, j_{l-1}, -j_l, j_{l+1}, \ldots, j_{(p-1)/2}\} \in \mathcal{U}.$$

We assume that $1 \leqslant n \leqslant (p-1)/2$ exists such that for each set $\mathcal{I} \in \mathcal{U}$ the coefficient a_n of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ is zero. It is impossible that $n = (p-1)/2$

because $a_{(p-1)/2}$ is the product of the $j \in \mathcal{I}$ and therefore nonzero. Now let $n \neq 0$, $n \neq (p-1)/2$; then we define for any $l = 1, \dots, (p-1)/2$

$$b_n^l := \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J|=n}} \prod_{j \in J} j, \qquad b_0^l := 1.$$

Then the coefficient of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ is $a_n = b_n^l + j_l b_{n-1}^l$. Because of our assumption, the coefficients of x^n in $c(\widetilde{\rho}_{\mathcal{I}})$ and in $c(\widetilde{\rho}_{\mathcal{I}_l})$ are $b_n^l + j_l b_{n-1}^l = 0$ and $b_n^l - j_l b_{n-1}^l = 0$ respectively. This implies that $b_n^l = 0$, $b_{n-1}^l = 0$ and

$$\begin{aligned} a_{n+1} &= \sum_{\substack{J \subseteq \mathcal{I} \\ |J|=n+1}} \prod_{j \in J} j \\ &= \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} \Big(j_l \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J|=n}} \prod_{j \in J} j \Big) = \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} j_l \, b_n^l = 0 . \end{aligned}$$

The factor $1/(n+1)$ appears because in the second line we have $n+1$ times each term appearing in the sum of the first line. Therefore $a_{n+1} = 0$ for each set $\mathcal{I} \in \mathcal{U}$, and by induction we get $a_{(p-1)/2} = 0$ for each set $\mathcal{I} \in \mathcal{U}$, which is impossible. □

Let $\mathrm{Sp}(\mathbf{Z}) := \bigcup_{n \geqslant 1} \mathrm{Sp}(2n, \mathbf{Z})$.

THEOREM 2.2. *For every $j \geqslant 1$, $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$ has infinite order.*

Proof. This theorem is a corollary of Theorem 2.1. A consequence of the stability result stated in section 2.1 is that for $p - 1 > 8j + 8$ the inclusion

$$\mathrm{Sp}(p-1, \mathbf{Z}) \longrightarrow \mathrm{Sp}(\mathbf{Z})$$

induces an isomorphism

$$\mathrm{H}^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} \mathrm{H}^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}).$$

In Theorem 2.1 we have shown that for any odd prime p and any integer $j = 1, \dots, (p-1)/2$ a representation $\widetilde{\rho}_{\mathcal{I}} \colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{U}((p-1)/2)$ exists that factors through $\mathrm{Sp}(p-1, \mathbf{Z})$ and for which the j-th Chern class $c_j(\widetilde{\rho}_{\mathcal{I}})$ is nonzero. Then the j-th symplectic class $d_j(\rho_{\mathcal{I}})$ is also nonzero. Here the representation $\rho_{\mathcal{I}}\colon \mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1,\mathbf{Z})$ is the one corresponding to $\widetilde{\rho}_{\mathcal{I}}$. We have an induced homomorphism

$$\begin{aligned} \rho_{\mathcal{I}}^*\colon \mathrm{H}^{2j}(\mathrm{Sp}(p-1,\mathbf{Z}),\mathbf{Z}) &\longrightarrow \mathrm{H}^{2j}(\mathbf{Z}/p\mathbf{Z},\mathbf{Z}) \\ d_j(\mathbf{Z}) &\longmapsto d_j(\rho_{\mathcal{I}}) . \end{aligned}$$

Herewith for any p the class $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(p-1,\mathbf{Z}),\mathbf{Z})$ is nonzero and has either infinite order or finite order divisible by p, since it restricts non-trivially to $\mathrm{H}^{2j}(\mathbf{Z}/p\mathbf{Z},\mathbf{Z})$. This shows that $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(\mathbf{Z}),\mathbf{Z})$ has infinite order. □

This is a new proof of a result of A. Borel [3]. He proved that $\mathrm{H}^*(\mathrm{Sp}(\mathbf{Z}),\mathbf{Q}) = \mathbf{Q}[d_1,d_3,\dots]$. Moreover, each d_{2i} can be expressed as a polynomial in the d_{2j+1}'s. This implies that all the $d_i(\mathbf{Z})$'s have infinite order.

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