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Artikel: THE POSITIVE CONE OF SPHERES AND SOME PRODUCTS OF SPHERES
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that for spaces "with only one high-dimensional cell" the γ-cone is "blind" in some sense to be made precise there.

Section 11 is devoted to explicitly computing the γ-operations for the products $S^{2n} \times S^{2m}$. As a consequence of these calculations, we establish a "doubling-formula" for Stirling numbers of the second kind. Moreover, we are led to conjecture that the same formula holds for Stirling numbers of the first kind. (This has now been proved by Al Lundell; see Theorem 11.2.)

2. PRELIMINARIES

We start by reviewing some topological K-theory. Our basic references are the books by Atiyah [Atiyah] and by Husemoller [Huse].

Let X be a *connected finite* CW-complex. (We assume all spaces and maps to be pointed.) For each $n \geq 0$, let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of complex n-plane vector bundles over X, and $\mathrm{Vect}(X)$ their disjoint union. There are well-known bijections

$$\mathrm{Vect}_n(X) \approx [X, BU(n)] \qquad (n \geq 0)$$

where $BU(n)$ is the classifying space of the unitary group $U(n)$ and $[\,.\,,\,.\,]$ stands for the set of homotopy classes of maps. For an n-plane vector bundle ξ over X, i.e. $\xi \in \mathrm{Vect}_n(X)$, we write $\mathrm{rk}(\xi) = n$ (it is the *rank* of ξ). The direct sum (also called Whitney sum) and the tensor product of vector bundles endow $\mathrm{Vect}(X)$ with a semiring structure. The K-theory of X is the ring $K(X)$, also denoted by $K^0(X)$, obtained by applying the Grothendieck construction to $\mathrm{Vect}(X)$, i.e. $K(X) = \mathcal{G}(\mathrm{Vect}(X))$. An element of $K(X)$ is sometimes called a *virtual vector bundle*. There is a ring isomorphism

$$K(X) \cong [X, \mathbf{Z} \times BU],$$

where BU is the infinite Grassmannian, i.e. the direct limit of the classifying spaces $BU(n)$. We identify both rings from now on. There is a canonical splitting $K(X) = \mathbf{Z} \oplus [X, BU] = \mathbf{Z} \oplus \widetilde{K}(X)$, where $\widetilde{K}(X) = \widetilde{K}^0(X)$ is the subring of *stable classes* of vector bundles, and $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$ is represented by the n-dimensional trivial vector bundle over X. Clearly, the Grothendieck construction gives rise to maps $\theta\colon \mathrm{Vect}(X) \longrightarrow K(X)$ and $\theta_n\colon \mathrm{Vect}_n(X) \longrightarrow n \times \widetilde{K}(X)$ (by restriction of θ).

DEFINITION 2.1.

i) The *positive cone* of X, denoted by $K_+(X)$, is the image of θ. An element $\xi \in K(X)$ is called *positive* if it lies in the positive cone.

ii) The *geometric dimension* of $x \in \widetilde{K}(X)$, denoted by g-dim(x), is the smallest integer n such that (n, x) lies in the image of θ_n, i.e. the least integer n such that the stable class x is represented by an n-dimensional vector bundle.

Since θ is a semiring homomorphism, it is clear that $K_+(X)$ is a subsemiring of $K(X)$. Notice that it is equivalent to determine the positive cone or the map g-dim: $\widetilde{K}(X) \longrightarrow \mathbf{Z}$; in fact, we have

$$K_+(X) = \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid n \geq \text{g-dim}(x)\}\,.$$

Let us also notice that an element $x \in \widetilde{K}(X)$, considered as a homotopy class of maps $X \longrightarrow BU$, has geometric dimension $\leq n$ if and only if x has a lifting $x_n \colon X \longrightarrow BU(n)$, i.e.

$$\begin{array}{ccc} & & BU(n) \\ & \overset{x_n}{\nearrow} & \downarrow{\scriptstyle i_n} \\ X & \xrightarrow{\;x\;} & BU \end{array} \qquad\qquad i_n \circ x_n = x$$

(Here, we identify a map with the homotopy class it represents.) Recall that i_n is a fibration with fibre $U/U(n)$, where $U = \varinjlim U(n)$ is the infinite unitary group (and BU is really its classifying space). The image of θ_n is equal to the image of the composition

$$[X, BU(n)] \xrightarrow{(i_n)_*} [X, BU] \hookrightarrow n \times [X, BU]\,, \qquad y \longmapsto (n, (i_n)_*(y))\,.$$

We write $K^*(X) = K^0(X) \oplus K^1(X)$, where the K^1-group is defined by

$$K^1(X) := [X, U]\,.$$

For a pair of connected finite CW-complexes (X, Y), there is the famous six-term exact sequence:

$$\begin{array}{ccccc} \widetilde{K}^0(X/Y) & \xrightarrow{q^*} & \widetilde{K}^0(X) & \xrightarrow{i^*} & \widetilde{K}^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \xleftarrow{i^*} & K^1(X) & \xleftarrow{q^*} & K^1(X/Y)\,, \end{array}$$

where $i \colon Y \hookrightarrow X$ is the inclusion and $q \colon X \twoheadrightarrow X/Y$ is the quotient map.

The n-th exterior power operation for complex vector spaces induces an operation on vector bundles denoted by $\xi \longmapsto \lambda^n\xi$, and endows $K(X)$ with a natural λ-ring structure. For $\xi \in K(X)$, one defines

$$\lambda_t(\xi) := \sum_{n\geq 0}(\lambda^n\xi)\cdot t^n \in K(X)[[t]]$$

(the latter being the ring of formal power series with coefficients in $K(X)$). The function λ_t is exponential, i.e. $\lambda_t(\xi+\eta) = \lambda_t(\xi)\cdot\lambda_t(\eta)$. Associated to the *$\lambda$-operations* are the *$\gamma$-operations* or *Grothendieck operations* $\gamma^n(\xi)$, which are defined by their generating series as follows:

$$\sum_{n\geq 0}\gamma^n(\xi)\cdot t^n = \gamma_t(\xi) := \lambda_{t/1-t}(\xi).$$

In particular, $\gamma^0(\xi) = 1$ and $\gamma^1(\xi) = \xi$. Again, the function γ_t is exponential, which implies that

$$\gamma^n(\xi+\eta) = \sum_{k=0}^{n}\gamma^k(\xi)\cdot\gamma^{n-k}(\eta).$$

The importance of the γ-operations in our context is illustrated by the following fact (see [Atiyah], Proposition 3.1.1):

Let $x \in \widetilde{K}(X)$; if g-dim$(x) \leq n$, then $\gamma^k(x) = 0$ for $k > n$.

(Assume that $(n, x) \in \mathbf{Z}\oplus\widetilde{K}(X)$ is represented by an n-dimensional vector bundle ξ. Then $\lambda_s(\xi)$ is a polynomial of degree n in s. By the exponential property, $\lambda_s(\xi) = \lambda_s(1)^n\cdot\lambda_s(x) = (1+s)^n\cdot\lambda_s(x)$. Letting $s = t/(1-t)$, we see that $\gamma_t(x) = \lambda_{t/1-t}(x) = (1-t)^n\lambda_{t/1-t}(\xi)$ is a polynomial of degree $\leq n$ in t.)

The representable K-theory of $BU(n)$, i.e. $[BU(n), \mathbf{Z}\times BU]$, is

$$K(BU(n)) = \mathbf{Z}[[\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^n]],$$

where $\widetilde{\gamma}^k = \gamma^k(\widetilde{\rho}_n)$, for $1 \leq k \leq n$, $\widetilde{\rho}_n$ being the stable class of the universal n-plane bundle ρ_n over $BU(n)$. Note that $\gamma^k(\widetilde{\rho}_n) = 0$, for all $k > n$, and that the map j^*, induced by $j\colon BU(n) \longrightarrow BU(n+l)$ in K-theory, takes $\widetilde{\rho}_{n+l}$ to $\widetilde{\rho}_n$, for any $l \geq 0$.

For a complex vector bundle ξ over X, the *n-th Chern class* $c_n(\xi)$ is a $2n$-dimensional integral cohomology class of X, i.e. $c_n(\xi) \in H^{2n}(X; \mathbf{Z})$. One has $c_0(\xi) = 1$, and the element $c(\xi) = \sum_{n\geq 0} c_n(\xi) \in H^*(X; \mathbf{Z})$, called the *total Chern class*, is exponential, i.e. it satisfies

$$c(\xi+\eta) = c(\xi)\cdot c(\eta).$$

The basic properties of Chern classes (see [Huse]) imply the following facts:

i) Two stably equivalent bundles over X have the same Chern classes. In particular, for an element $x \in \widetilde{K}(X)$, the n-th Chern class $c_n(x) \in H^{2n}(X; \mathbf{Z})$ is well-defined.

ii) If $n > \mathrm{rk}(\xi)$, then $c_n(\xi) = 0$.

iii) Let $x \in \widetilde{K}(X)$; if g-dim$(x) \leq n$, then $c_k(x) = 0$ for $k > n$.

Let us also formally define the polynomial

$$c_\xi(t) := \sum_{n \geq 0} c_n(\xi) \cdot t^n \in H^*(X; \mathbf{Z})[t],$$

which, by ii) above, is indeed a polynomial. It is also exponential.

A central feature of Chern classes is that the cohomology ring of $BU(n)$ is given by

$$H^*(BU(n); \mathbf{Z}) = \mathbf{Z}[\widetilde{c}_1, \ldots, \widetilde{c}_n],$$

where $\widetilde{c}_k = c_k(\widetilde{\rho}_n)$, for $1 \leq k \leq n$. Moreover, $c_k(\widetilde{\rho}_n) = 0$, for any $k > n$. On the combinatorial point of view, for any $n \geq 0$, the Grassmannians $BU(n)$ and BU admit CW-decompositions with the same $(2n+1)$-skeleton, in other words, such that $BU(n)^{[2n+1]} = BU^{[2n+1]}$. (This can be proved by adapting Section 6 of [MilSt] to the complex case.)

The Chern character ch is a multiplicative natural transformation from K-theory to rational cohomology

$$ch\colon K(X) \longrightarrow H^{ev}(X; \mathbf{Q}) = \bigoplus_{q \geq 0} H^{2q}(X; \mathbf{Q}), \quad \xi \longmapsto ch(\xi) = \sum_{q \geq 0} ch_{2q}(\xi),$$

where $ch_{2q}(\xi) \in H^{2q}(X; \mathbf{Q})$ (X being a connected *finite* CW-complex). It relates γ-operations and Chern classes as given in the following well-known proposition. Before stating it, we introduce some notation. For $x \in \widetilde{K}(X)$, we let $\bar{c}_j(x)$ be the image of $c_j(x)$ under the coefficient homomorphism $H^{2j}(X; \mathbf{Z}) \longrightarrow H^{2j}(X; \mathbf{Q})$, and let $I_x(\bar{c}_k, \ldots, \bar{c}_n)$ be the ideal in $H^{ev}(X; \mathbf{Q})$ generated by $\bar{c}_k(x), \ldots, \bar{c}_n(x)$, where $x \in \widetilde{K}(X)$ and $n \geq k$.

PROPOSITION 2.2. *Let X be a connected finite CW-complex of dimension $\leq 2n$, $x \in \widetilde{K}(X)$, and $k \leq n$. One then has*

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \ldots, \bar{c}_n(x)),$$

where P_{k+1} is a polynomial in $\bar{c}_1(x), \ldots, \bar{c}_n(x)$ contained in the ideal

$$I_x(\bar{c}_k, \ldots, \bar{c}_n) \cap \Big(\bigoplus_{q \geq k+1} H^{2q}(X; \mathbf{Q}) \Big).$$

In particular,

$$ch(\gamma^n(x)) = \bar{c}_n(x) \in H^{2n}(X; \mathbf{Q}) .$$

Proof. For a line bundle $\eta = 1 + y$, one has $ch(\gamma_t(y)) = 1 + (e^{c_1(y)} - 1) \cdot t$ in $H^{ev}(X; \mathbf{Q})[[t]]$, therefore, the result follows readily for the Whitney sum of n universal line bundles over $\mathbf{C}P^\infty \times \ldots \times \mathbf{C}P^\infty$ (n factors) and hence for any element x represented by a Whitney sum of line bundles. The general case is obtained by invoking the splitting principle. □

(Proposition 2.2 shows that $\gamma^k(x)$ is of filtration $\geq k$, in the terminology of Atiyah and Hirzebruch [AtHi].)

Recall that a finite CW-complex is called *torsion-free* if its integral homology (or equivalently integral cohomology) contains no torsion. The Chern classes give some "sharp information" on the geometric dimension, as the next fundamental theorem shows.

THEOREM 2.3. *Consider a connected finite CW-complex X of dimension $\leq 2n$, and $x \in \widetilde{K}(X)$. Then*

$$\text{g-dim}(x) < n \iff c_n(x) = 0 .$$

If moreover X is torsion-free, then this is also equivalent to $\gamma^n(x) = 0$.

Proof. If $\text{g-dim}(x) < n$, as already mentioned, $c_n(x) = 0$. The converse is a consequence of Theorem 41.5 on page 210 of [Steen] (the Chern classes of a vector bundle are the same as the Chern classes of the associated spherical bundle as defined in [Steen]). The last statement follows from Proposition 2.2 and injectivity of the Chern character for a torsion-free space. □

Let us recall the K-theory of the spheres:

$$K^0(S^{2n}) \cong \mathbf{Z} \oplus \mathbf{Z} \qquad K^0(S^{2n+1}) \cong \mathbf{Z}$$
$$K^1(S^{2n}) = 0 \qquad K^1(S^{2n+1}) \cong \mathbf{Z} .$$

(For technical reasons, we will always implicitly exclude the 0-sphere.) The multiplicative structure on $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x_{2n}$ is given by $x_{2n}^2 = 0$. The γ-operations and the Chern classes are as given in the next proposition.

PROPOSITION 2.4. *Let x_{2n} be a generator of $\widetilde{K}(S^{2n}) \cong \mathbf{Z}$. Then*

i) $\gamma^k(x_{2n}) = (-1)^{k-1}(k-1)!\, S(n,k) \cdot x_{2n}$, where $S(n,k)$ is a Stirling number of the second kind.

ii) $c_n(x_{2n}) = (-1)^{n-1}(n-1)! \cdot a_{2n}$, where a_{2n} is a suitable generator of the cohomology group $H^{2n}(S^{2n}; \mathbf{Z}) \cong \mathbf{Z}$.

Proof. It is well-known that $\lambda^k(x_{2n}) = (-1)^{k+1}k^{n-1} \cdot x_{2n}$, for $k \geq 1$ (see Proposition 2.5 and Theorem 11.2 in Chapter 13 of [Huse]). We thus get

$$\begin{aligned}
\gamma_t(x_{2n}) = \lambda_{t/1-t}(x_{2n}) &= 1 + \Big(\sum_{k\geq 1}(-1)^{k+1}k^{n-1}t^k(1-t)^{-k}\Big) \cdot x_{2n} \\
&= 1 - \Big(\sum_{k\geq 1}\sum_{j\geq 0}(-1)^{k+j}\tbinom{-k}{j}k^{n-1} \cdot t^{k+j}\Big) \cdot x_{2n} \\
&= 1 - \Big(\sum_{k\geq 1}\sum_{j\geq 0}(-1)^{k}\tbinom{k+j}{k}\tfrac{k^n}{k+j} \cdot t^{k+j}\Big) \cdot x_{2n} \\
&\overset{(*)}{=} 1 - \Big(\sum_{m\geq 1}\Big(\sum_{k=1}^{m}(-1)^{k}\tbinom{m}{k}\tfrac{k^n}{m}\Big) \cdot t^{m}\Big) \cdot x_{2n} \\
&= 1 + \Big(\sum_{m\geq 1}(-1)^{m-1}(m-1)!\, S(n,m) \cdot t^{m}\Big) \cdot x_{2n}\,,
\end{aligned}$$

since $S(n,m) = \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\frac{k^n}{m!}$ (see (6.19) on page 251 of [GKP]), hence the first formula. (The equality $(*)$ is obtained by substituting $m := k+j$.)

The second formula follows from Theorem 9.6 and Corollary 9.8 (and its proof) in Chapter 20 of [Huse]. □

Let us finally state a lemma relating γ-operations and Chern classes. We shall need it further on.

LEMMA 2.5. *Let Y be a connected CW-complex (possibly infinite). Then, for an element $x \in \widetilde{K}(Y) = [Y, BU]$, one has*

$$c_n(\gamma^n(x)) = (-1)^{n-1}(n-1)!\, c_n(x) \in H^{2n}(Y; \mathbf{Z}).$$

Proof. Let $i\colon BU(n-1) \longrightarrow BU$ be the canonical map, and

$$i^*\colon H^*(BU;\mathbf{Z}) = \mathbf{Z}[\widetilde{c}_1, \widetilde{c}_2, \ldots] \longrightarrow \mathbf{Z}[\widetilde{c}_1, \ldots, \widetilde{c}_{n-1}] = H^*(BU(n-1);\mathbf{Z})$$

the induced map. Since $c_n(\widetilde{\gamma}^n) \in \mathrm{Ker}(i^*) \cap H^{2n}(BU;\mathbf{Z}) = \mathbf{Z} \cdot \widetilde{c}_n$, there exists an integer q_n such that $c_n(\widetilde{\gamma}^n) = q_n \cdot \widetilde{c}_n$. Recalling that $S(n,n) = 1$, an easy computation (based on Proposition 2.4) for the sphere S^{2n} shows that one has $q_n = (-1)^{n-1}(n-1)!$, as claimed. □

3. THE γ-CONE AND THE c-CONE

In general, the problem of computing the geometric dimension of vector bundles is very complicated, as is any general lifting problem in homotopy theory. So, the same is true for the positive cone. That is why we now introduce what we call the γ-cone and the c-cone. They are supposed to be easier to compute and might be good approximations to the positive cone. As we will see, these two cones coincide for torsion-free spaces.

DEFINITION 3.1.

i) The *γ-cone* of X is defined by

$$K_\gamma(X) := \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid \gamma^k(x) = 0 \text{ for all } k > n\}.$$

The *γ-dimension* of a class $x \in \widetilde{K}(X)$, denoted by $\gamma\text{-dim}(x)$, is the least integer n such that $\gamma^k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $\gamma_t(x)$.

ii) The *c-cone* of X is defined by

$$K_c(X) := \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid c_k(x) = 0 \text{ for all } k > n\}.$$

The *c-dimension* of a class $x \in \widetilde{K}(X)$, denoted by $\text{c-dim}(x)$, is the least integer n such that $c_k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $c_x(t)$.

Let us point out that the "lower boundary" of the positive cone $K_+(X)$, as a subset of $\widetilde{K}(X) \oplus \mathbf{Z}$, coincides with the graph of the geometric dimension function $\text{g-dim}\colon \widetilde{K}(X) \longrightarrow \mathbf{Z}$ (the positive elements consisting exactly of the boundary and the points located above it). The analogous statements hold for the γ-cone and the c-cone with respect to the corresponding dimension function.

The following results on these objects follow readily from our preliminaries on K-theory.