5. FURTHER PROPERTIES OF THE CONES

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LEMMA 4.2. *For $m \geq 2k+1$, the group $\pi_m(BU(k))$ is finite.*

Proof. We fix $m \geq 3$. The fibration $BU(k-1) \longrightarrow BU(k)$, with fibre S^{2k-1}, yields the following long exact sequence in homotopy:

$$\ldots \to \pi_m(S^{2k-1}) \to \pi_m(BU(k-1)) \to \pi_m(BU(k)) \to \pi_{m-1}(S^{2k-1}) \to \ldots$$

By Serre [Serre], $\pi_j(S^{2k-1})$ is finite for $j \neq 2k-1$, and we can conclude by induction over k (with $k \geq 1$ and $2k+1 \leq m$), since when $k=1$, one has $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$ for $m \geq 3$. □

From this, we now infer that the image of $(i_k)_*$ is zero for $k < n$. This implies that g-dim$(lx) = n$ when $l \neq 0$, and concludes the second proof.

REMARK 4.3.

i) Since we were motivated by Elliott's classification of unital C^*-algebras of type AF by means of their K-theory, their positive cone and the K-theory class [1] of the unit (see [Bla1]), it is important to single out the fact that the positive cone of S^{2n} and that of S^{2m} are non-isomorphic as monoids if n is different from m. (There is no need here to distinguish the K-theory class 1 of the trivial one-dimensional bundle.) Let us provide a short proof of this claim. For $n \geq 1$, let M_n denote the positive cone of S^{2n} (identified as above with a sub-monoid of $\mathbf{Z}^2$, in order to designate its elements). The abelian monoid M_n has a minimal set A_n of generators, in other words a generating set (as a monoid) that is contained in any other generating set, namely

$$A_n = \{(0,1)\} \cup \{(k,n) \mid k \in \mathbf{Z} \setminus \{0\}\}.$$

Now, consider the function $\sigma \colon A_n \longrightarrow \{2, 3, \ldots\}$ defined, for $x \in A_n$, by

$$\sigma(x) := \min \{l \geq 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}.$$

Clearly, such an l exists for any $x \in A_n$ and $\sigma(A_n) = \{2, 2n\}$. Since A_n and σ are isomorphism invariants of M_n, this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial"; in other words, $K(S^{2n-1}) = \mathbf{Z}$ and $K_+(S^{2n-1}) = \mathbf{N}$.

5. FURTHER PROPERTIES OF THE CONES

We now investigate naturality properties and behaviour under products of the positive cone, the γ-cone and the c-cone.

The following result is obvious.

PROPOSITION 5.1. *Let $f: X \longrightarrow Y$ be a map between connected finite CW-complexes. Let $f^*: K(Y) \longrightarrow K(X)$ be the λ-homomorphism induced by f. Then, for any $y \in \widetilde{K}(Y)$, one has*

$$\text{g-dim}(f^*(y)) \leq \text{g-dim}(y)$$

$$\gamma\text{-dim}(f^*(y)) \leq \gamma\text{-dim}(y)$$

$$\text{c-dim}(f^*(y)) \leq \text{c-dim}(y)\,,$$

and in particular,

$$f^*(K_+(Y)) \subseteq K_+(X)$$

$$f^*(K_\gamma(Y)) \subseteq K_\gamma(X)$$

$$f^*(K_c(Y)) \subseteq K_c(X)\,.$$

Furthermore, if f^* is an isomorphism, then

$$f^*(K_\gamma(Y)) = K_\gamma(X)\,.$$

For the next corollary we need a new definition.

DEFINITION 5.2. Let X and Y be two connected finite CW-complexes. A map $f: X \longrightarrow Y$ is called a *K^0-equivalence* (or *K-equivalence* for short) if there exists a map $g: Y \longrightarrow X$ such that, on the level of the K^0-groups,

$$f^* \circ g^* = Id_{K^0(X)} \quad \text{and} \quad g^* \circ f^* = Id_{K^0(Y)}\,.$$

Note that a K-equivalence is *not* necessarily a homotopy equivalence: there are homotopically non-trivial (i.e. non-contractible) finite CW-complexes X for which $\widetilde{K}(X) = 0 = \widetilde{K}(pt)$; see example iii) below.

PROPOSITION 5.3. *If $f: X \longrightarrow Y$ is a K-equivalence, then f induces the following isomorphisms of semigroups:*

$$K_+(Y) \overset{f^*}{\cong} K_+(X) \quad \text{and} \quad K_\gamma(Y) \overset{f^*}{\cong} K_\gamma(X)\,.$$

Proof. Applying Proposition 5.1 twice, we get (in the notations of Definition 5.2)

$$K_+(X) = f^* \circ g^*(K_+(X)) \subseteq f^*(K_+(Y)) \subseteq K_+(X)\,.$$

This establishes the first isomorphism, whereas the second one is obvious. □

The following result is more technical to state.

COROLLARY 5.4. *Let X and Y be two connected finite CW-complexes. Assume that $K^1(X) = 0$ and that $\widetilde{K}^0(Y) = 0$. Then the projection $p\colon X \times Y \longrightarrow X$ induces isomorphisms*

$$K_+(X) \overset{p^*}{\cong} K_+(X \times Y) \quad \textit{and} \quad K_\gamma(X) \overset{p^*}{\cong} K_\gamma(X \times Y)\,.$$

Proof. Invoking the Künneth theorem for K-theory, our hypotheses imply that $p^*\colon K^0(X) \longrightarrow K^0(X \times Y)$ is an isomorphism with inverse i^*, where i is the inclusion of X in $X \times Y$. Consequently, p^* is a K-equivalence. □

The following is a useful result.

PROPOSITION 5.5. *Let X and Y be connected finite CW-complexes. Assume that the positive cone and the γ-cone of Y coincide, and let $f\colon X \longrightarrow Y$ be a map inducing an isomorphism $f^*\colon K(Y) \longrightarrow K(X)$. Then f induces an isomorphism of positive cones, and the γ-cone of X coincides with the positive cone:*

$$K_+(Y) \overset{f^*}{\cong} K_+(X) = K_\gamma(X)\,.$$

Proof. By Proposition 5.1 we have $f^*(K_+(Y)) = f^*(K_\gamma(Y)) = K_\gamma(X)$ and $f^*(K_+(Y)) \subseteq K_+(X)$, hence $K_\gamma(X) \subseteq K_+(X)$. We conclude with iii) of Proposition 3.2. □

EXAMPLES.

i) Let X be a connected finite CW-complex of dimension ≤ 3. Since for suitable CW-decompositions, one has $BU(1)^{[3]} = BU^{[3]}$ and since $BU(1) = \mathbf{C}P^\infty = K(\mathbf{Z}, 2)$, any $x \in \widetilde{K}(X) = [X, BU]$ lifts to a class in $[X, BU(1)]$, giving an isomorphism $\widetilde{K}(X) \cong H^2(X; \mathbf{Z})$ mapping x to $c_1(x)$. It follows that the positive cone coincides with the c-cone and is given by

$$K_+(X) = \mathbf{N} \times \{0\} \ \cup\ \mathbf{N}^* \times \widetilde{K}(X) \subset \mathbf{Z} \times \widetilde{K}(X)\,.$$

ii) Example i) applies to a closed oriented surface Σ_g of genus g. Since it is torsion-free, its positive cone coincides with its c-cone and with its γ-cone. Moreover, let $f\colon \Sigma_g \longrightarrow S^2$ be a map of degree 1 (it exists, since both the 2-sphere and Σ_g are quotients of the square $[0, 1]^2$). Then f not only induces an isomorphism in K-theory, but also an isomorphism of positive cones, as follows from Proposition 5.1.

iii) Let X and Y be the Moore spaces $M(\mathbf{Z}/3, 2q+11) = S^{2q+11} \cup_3 e^{2q+12}$ and $M(\mathbf{Z}/3, 2q-1) = S^{2q-1} \cup_3 e^{2q}$ respectively. In [Adams], Adams shows that for q large enough, there exists a map $A\colon X = \Sigma^{12} Y \longrightarrow Y$ such that the induced map $A^*\colon \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$ is an isomorphism (take $p = m = 3$, $f = 1$ and $r = 6$ in Theorem 1.7 and in Lemmas 12.4 and 12.5 of [Adams]). Therefore, A is a K-isomorphism between simply connected finite CW-complexes, but it is *not* a homotopy equivalence. The mapping cone C_A is a non-contractible finite CW-complex with $\widetilde{K}(C_A) = 0$. (It is non-contractible because its homology is non-trivial.)

iv) In [GrMo], pp. 203-206, a CW-complex $X = (S^1 \vee S^2) \cup e^3$ is defined, with the property that the inclusion $i\colon S^1 = X^{[1]} \hookrightarrow X$ of the 1-skeleton induces an isomorphism in integral homology (and on the level on fundamental groups); however, i is *not* a homotopy equivalence since $\pi_2(X) \neq 0$. Consequently, by the universal coefficient theorem (see Corollary V.7.2 in [Bred]), i induces an isomorphism in integral cohomology, and, by a direct application of the Atiyah-Hirzebruch spectral sequence, also in K-theory. In particular, i is a K-equivalence, but *not* an equivalence. (As C_A in the preceding example, the quotient space $X/X^{[1]}$ has vanishing $\widetilde{K}$, however it is the closed 3-ball and is therefore contractible.)

Let us finally mention that in [Matt], the positive cone, the c-cone and the γ-cone are also studied from the rational point of view, and rational K-theory is considered.

6. The cones of the products $S^n \times S^{2m-1}$

In this section, we will compute the cones for the products $S^{2n} \times S^{2m-1}$ and $S^{2n-1} \times S^{2m-1}$.

We begin with $S^{2n} \times S^{2m-1}$. Since $\widetilde{K}(S^{2m-1}) = 0$ and $K^1(S^{2n}) = 0$, the answer immediately follows from Proposition 5.5.

THEOREM 6.1. *The projection* $p\colon S^{2n} \times S^{2m-1} \longrightarrow S^{2n}$ *induces an isomorphism of positive cones, and, for* $S^{2n} \times S^{2m-1}$, *the* γ*-cone and the* c*-cone coincide with the positive cone:*

$$K_+(S^{2n}) \overset{p^*}{\cong} K_+(S^{2n} \times S^{2m-1}) = K_\gamma(S^{2n} \times S^{2m-1}).$$