# PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

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# PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

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ABSTRACT. This paper is concerned with discrete versions of three well-known results from projective differential geometry: the four-vertex theorem, the theorem on six affine vertices, and Ghys' theorem on four zeroes of the Schwarzian derivative. We study the geometry of closed polygonal lines in $\mathbf{RP}^d$ and prove that polygons satisfying a certain convexity condition have at least $d+1$ flattenings. This result provides a new approach to the classical theorems mentioned.

## 1. INTRODUCTION

A *vertex* of a smooth plane curve is a point where the curve has $4^{\text{th}}$-order contact with a circle (at a generic point the osculating circle has $3^{\text{rd}}$-order contact with the curve). An *affine vertex* (or *sextactic point*) of a smooth plane curve is a point of $6^{\text{th}}$-order contact with a conic. In 1909 S. Mukhopadhyaya [10] published two celebrated theorems, which are joined in the following statement:

*Every closed smooth convex plane curve has at least* 4 *distinct vertices and at least* 6 *distinct affine vertices.*

These results generated an extensive literature. From a modern point of view they are related, among other subjects, to the global singularity theory of wave fronts and to Sturm theory – see e.g. [1, 2, 4, 8, 17, 18] and references therein.

A recent and unexpected result along these lines is the following theorem due to E. Ghys [7]:

*The Schwarzian derivative of every diffeomorphism of the projective line has at least 4 distinct zeroes.*

(see also [11, 15, 6]). The Schwarzian derivative vanishes when the $3^{rd}$ jet of the diffeomorphism coincides with that of a projective transformation (at a generic point a diffeomorphism can be approximated by a projective transformation up to the $2^{nd}$ derivative). Ghys' theorem can be interpreted as the 4-vertex theorem in Lorentzian geometry (cf. references above).

The goal of this note is to study polygonal analogues of the above results. In our opinion, such a discretization of smooth formulations is interesting for the following reasons. Firstly, a discrete theorem is a priori stronger; it becomes, in the limit, a smooth one, thus providing a new proof of the latter. An important feature of the discrete approach is the availability of mathematical induction, which can considerably simplify the proofs. Secondly, the very operation of discretization is non-trivial: a single smooth theorem may lead to non-equivalent discrete ones. An example of this phenomenon is provided by two recent versions of the 4-vertex theorem for convex plane polygons [12, 13, 19, 16] – see Remark 2.4 below. To the best of our knowledge, these results are the only available discrete versions of the 4-vertex theorem.

In this regard we would like to draw attention to a famous lemma of Cauchy (1813):

*Given two convex (plane or spherical) polygons whose respective sides are congruent, the cyclic sequence of the differences of respective angles of the polygons changes sign at least 4 times.*

This result plays a crucial role in the proof of rigidity for convex polyhedra (see [5] for a survey). The Cauchy lemma implies, in the limit, the smooth 4-vertex theorem and can be viewed as the first result in the area under discussion.

## 2. THEOREMS ON PLANE POLYGONS

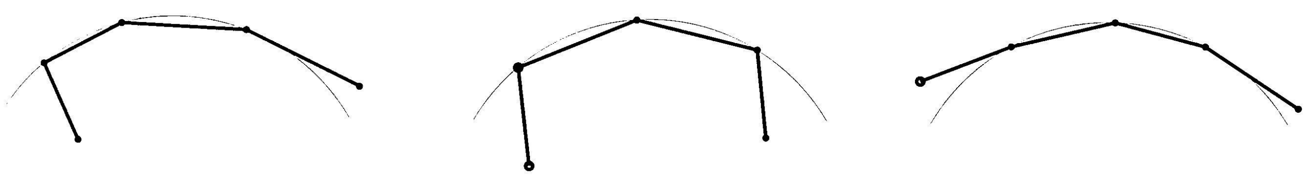
In this section we formulate our results for plane polygonal curves. The proofs will be given in Section 4.1.

### 2.1 DISCRETE 4-VERTEX THEOREM

The *osculating circle* of a smooth plane curve at a point is the circle (or straight line) that has 3rd order of contact with the curve at the given point. One may say that the osculating circle goes through 3 infinitely close points; at a vertex the osculating circle passes through 4 infinitely close points. Moreover, a generic curve crosses the osculating circle at a generic point and stays on one side of it at a vertex. This well-known fact motivates the following definition.

Let $P$ be a plane convex $n$-gon; throughout this section we assume that $n \geq 4$ . Denote the consecutive vertices by $V_1, \dots, V_n$ ; the subscripts are understood cyclically, that is, $V_{n+1} = V_1$ , etc.

DEFINITION 2.1. A triple of vertices $(V_i, V_{i+1}, V_{i+2})$ is said to be *extremal*[1]) if $V_{i-1}$ and $V_{i+3}$ lie on the same side of the circle through $V_i, V_{i+1}, V_{i+2}$ (this does not exclude the case where $V_{i-1}$ or $V_{i+3}$ belongs to the circle).



a) not extremal b) extremal

FIGURE 1

The next result follows from a somewhat more general theorem due to O. Musin and V. Sedykh [12] (see also [13]).

[1]) We have a terminological difficulty here: as we are dealing with polygons, we cannot use the term "vertex" in the same sense as in the smooth case; hence the term "extremal".

THEOREM 2.2. *Every plane convex polygon $P$ has at least 4 extremal triples of vertices.*

EXAMPLE 2.3. If $P$ is a quadrilateral then the theorem holds tautologically since the $(i-1)^{\text{st}}$ vertex coincides with the $(i+3)^{\text{rd}}$ for every $i$.

REMARK 2.4. An alternative approach to discretization of the 4-vertex theorem consists in inscribing circles in consecutive triples of sides of a polygon (the centre of such a circle is the intersection point of the bisectors of consecutive angles of the polygon). Then a triple of sides $(\ell_i, \ell_{i+1}, \ell_{i+2})$ is said to be *extremal* if the lines $\ell_{i-1}, \ell_{i+3}$ either both intersect the corresponding circle or both fail to intersect it. With this definition an analogue of Theorem 2.2 holds true [19, 16], and this, in the limit, also provides the smooth 4-vertex theorem.

Both formulations, concerning circumscribed or inscribed circles, make sense on the sphere. Moreover, they are equivalent via projective duality.

## 2.2 DISCRETE THEOREM ON 6 AFFINE VERTICES

Five generic points in the plane determine a conic. Considering the plane as an affine part of the projective plane, the complement of the conic has two connected components. Let $P$ be a plane convex $n$-gon; throughout this section we assume that $n \geq 6$. As in the previous section, we introduce the following definition.

DEFINITION 2.5. Five consecutive vertices $V_i, \ldots, V_{i+4}$ are said to be *extremal* if $V_{i-1}$ and $V_{i+5}$ lie on the same side of the conic through these 5 points (this does not exclude the case where $V_{i-1}$ or $V_{i+5}$ belongs to the conic).

If $P$ is replaced by a smooth convex curve, and $V_i, \ldots, V_{i+4}$ are infinitely close points, we recover the definition of an affine vertex. Hence the following theorem is a discrete version of the smooth theorem on 6 affine vertices.

THEOREM 2.6. *Every plane convex polygon $P$ has at least 6 extremal quintuples of vertices.*

EXAMPLE 2.7. If $P$ is a hexagon then the theorem holds tautologically for the same reason as in Example 2.3.

REMARK 2.8. On interchanging sides and vertices, and replacing circumscribed conics by inscribed ones, we arrive at a "dual" theorem. The latter is equivalent to Theorem 2.6 via projective duality – cf. Remark 2.4.

## 2.3 DISCRETE GHYS THEOREM

A discrete object of study in this section is a pair of cyclically ordered $n$-tuples $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ in $\mathbf{RP}^1$ with $n \geq 4$. We choose an orientation of $\mathbf{RP}^1$ and assume that the cyclic ordering of each of the two $n$-tuples is induced by this orientation.

Recall that an ordered quadruple of distinct points in $\mathbf{RP}^1$ determines a number, the *cross-ratio*, which is a projective invariant. Choosing an affine parameter such that the points are given by real numbers $a < b < c < d$, the cross-ratio is

$$[a, b, c, d] = \frac{(c-a)(d-b)}{(b-a)(d-c)}. \tag{2.1}$$

DEFINITION 2.9. A triple of consecutive indices $(i, i+1, i+2)$ is said to be *extremal* if the difference of cross-ratios

$$[y_j, y_{j+1}, y_{j+2}, y_{j+3}] - [x_j, x_{j+1}, x_{j+2}, x_{j+3}] \tag{2.2}$$

changes sign as $j$ varies from $i-1$ to $i$ (this does not exclude the case where either of the differences vanishes).

THEOREM 2.10. *For every pair $X, Y$ of $n$-tuples of points as above, there exist at least four extremal triples.*

EXAMPLE 2.11. If $n = 4$ then the theorem holds for a very simple reason. A cyclic permutation of four points induces the following transformation of their cross-ratio:

$$[x_4, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3, x_4]}{[x_1, x_2, x_3, x_4] - 1}, \tag{2.3}$$

and this is an involution. Furthermore, if $a > b > 1$ then $a/(a-1) < b/(b-1)$. Therefore, each triple of indices is extremal.

Let us interpret Theorem 2.10 in geometrical terms like Theorems 2.2 and 2.6. There exists a unique projective transformation that carries $x_i, x_{i+1}, x_{i+2}$ into $y_i, y_{i+1}, y_{i+2}$, respectively. The graph $G$ of this transformation can be seen as a curve in $\mathbf{RP}^1 \times \mathbf{RP}^1$ ; the three points $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$, $(x_{i+2}, y_{i+2})$ lie on this graph. An ordered pair of points $(x_j, x_{j+1})$ in oriented $\mathbf{RP}^1$ defines a unique segment. An ordered pair of points $\big((x_j, y_j), (x_{j+1}, y_{j+1})\big)$ in $\mathbf{RP}^1 \times \mathbf{RP}^1$ also defines a unique segment, namely the one whose projection on each factor is a segment in $\mathbf{RP}^1$ as defined before. The triple $(i, i+1, i+2)$ is extremal if and only if the topological intersection index of the broken line $(x_{i-1}, y_{i-1}), \dots, (x_{i+3}, y_{i+3})$ with the graph $G$ is zero. This fact can be checked from (2.1) by a direct computation, which we omit.

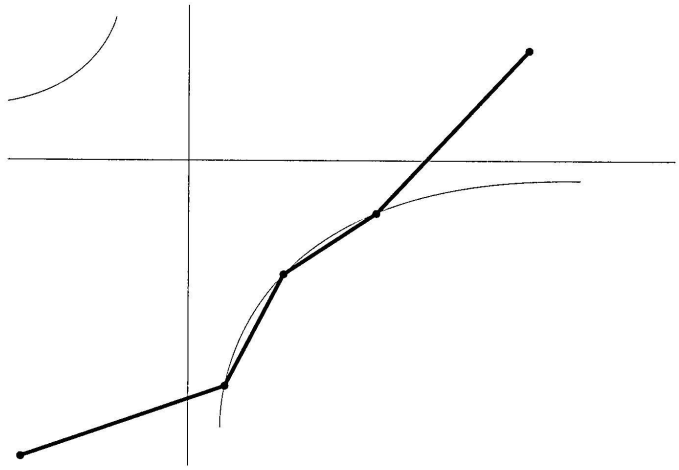


FIGURE 2

Let us also comment on the relation between Definition 2.9 and the zeroes of the Schwarzian derivative of a diffeomorphism of the projective line. Let

$$x_0 = 0, \quad x_1 = \varepsilon, \quad x_2 = 2\varepsilon, \quad x_3 = 3\varepsilon$$

be four infinitely close points given in some affine coordinate, and let $y_i = f(x_i)$ where $f$ is a diffeomorphism of $\mathbf{RP}^1$. Then a direct computation using (2.1) yields:

$$[y_0, y_1, y_2, y_3] - [x_0, x_1, x_2, x_3] = \varepsilon^2\, S(f)(0) + O(\varepsilon^3),$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of $f$. Thus, for $\varepsilon \to 0$, Definition 2.9 corresponds to the vanishing of the Schwarzian derivative.

## 3. Main Theorem

All theorems from Section 2 are consequences of one theorem on the least number of flattenings of a closed polygon in real projective space.

In his remarkable work [3], M. Barner introduced the notion of a *strictly convex* curve in real projective space: this is a smooth closed curve $\gamma \subset \mathbf{RP}^d$ such that for every $(d-1)$-tuple of points on $\gamma$ there exists a hyperplane through these points that does not intersect $\gamma$ at any other points. Barner discovered the following theorem:

*A strictly convex curve has at least $d+1$ distinct flattening points.*

Recall that a flattening point of a projective space curve is a point at which the osculating hyperplane is stationary; in other words, this is a singularity of the projectively dual curve. In fact, Barner's result is considerably stronger but we shall not dwell on it here – see [15] for an exposition.

Our goal in this section is to provide a discrete version of Barner's theorem. First we need to develop an elementary intersection formalism for polygonal lines.

### 3.1 Intersection multiplicities

Throughout this section we shall look at closed polygons $P \subset \mathbf{RP}^d$ with vertices $V_1, \dots, V_n$ $(n \geq d+1)$ in general position. In other words, for every set of vertices $V_{i_1}, \dots, V_{i_k}$, where $k \leq d+1$, the span of $V_{i_1}, \dots, V_{i_k}$ is $(k-1)$-dimensional.

Definition 3.1. A polygon $P$ is said to be *transverse* to a hyperplane $H$ at a point $X \in P \cap H$ if

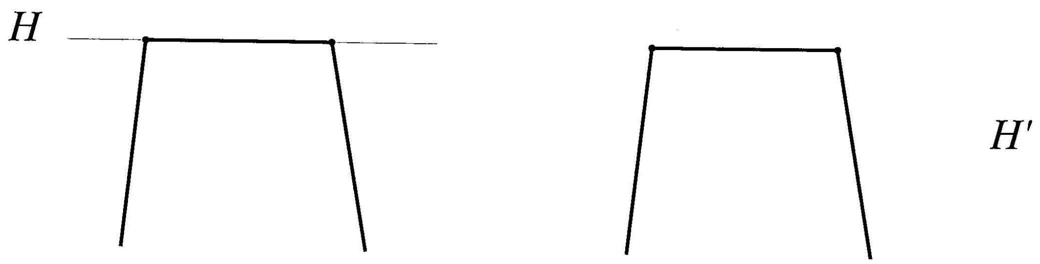
(a) $X$ is an interior point of an edge and this edge is transverse to $H$, or

(b) $X$ is a vertex, the two edges incident to $X$ are transverse to $H$ and are locally separated by $H$.

Clearly, transversality is an open condition.

Definition 3.2. A polygon $P$ is said to intersect a hyperplane $H$ with multiplicity $k$ if for every hyperplane $H'$ sufficiently close to $H$ and transverse to $P$, the number of points $P \cap H'$ does not exceed $k$ and, moreover, $k$ is attained for some $H'$.

This definition does not exclude the case where a number of vertices of $P$ lie in $H$.




multiplicity 2

FIGURE 3

LEMMA 3.3. *Let $V_{i_1}, \dots, V_{i_k}$ with $k \le d$ be vertices of $P$. Then any hyperplane $H$ passing through $V_{i_1}, \dots, V_{i_k}$ meets $P$ with multiplicity at least $k$.*

*Proof.* Move each $V_{i_j}$ $(j = 1, \dots, k)$ slightly along the edge $(V_{i_j}, V_{i_j+1})$ to obtain a new point $V'_{i_j}$. Let us show that a generic hyperplane $H'$ through $V'_{i_1}, \dots, V'_{i_k}$ is transverse to $P$. This will imply the lemma because $H'$ has at least $k$ intersections with $P$.

It suffices to show that $H'$ does not contain any vertex of $P$. First we note that, since $P$ is in general position, a generic hyperplane $H$ through $V_{i_1}, \dots, V_{i_k}$ does not contain any other vertex. The same holds true for every hyperplane which is sufficiently close to $H$. It remains to show that the chosen $H'$ does not contain any of $V_{i_1}, \dots, V_{i_k}$.

Suppose $H'$ contains $V_{i_j}$. Then $H'$ contains the edge $(V_{i_j}, V_{i_j+1})$ and therefore also $V_{i_j+1}$. If $i_j + 1 \notin \{i_1, \dots, i_k\}$ we obtain a contradiction with the previous paragraph. If, on the other hand, $i_j + 1 \in \{i_1, \dots, i_k\}$ then we can proceed in the same way with $V_{i_j+1}$. However, we cannot go on indefinitely since $k < n$. $\square$

The next definition is topological in nature.

DEFINITION 3.4. Consider a continuous curve in $\mathbf{RP}^d$ with endpoints $A$ and $Z$. Let $H$ be a hyperplane not containing $A$ or $Z$. We say that $A$ and $Z$ *are on one side of $H$* if one can connect $A$ and $Z$ by a curve not intersecting $H$ in such a way that the resulting closed curve is contractible. Otherwise we say that $A$ and $Z$ are *separated by $H$*.

Clearly, if one has only two points $A$ and $Z$ (and no curve connecting them), then one cannot say that the points are on one side of, or separated by, a hyperplane.

LEMMA 3.5. *Let $\Gamma = (A, \dots, Z)$ be a broken line in general position in $\mathbf{RP}^d$, and let $H$ be a hyperplane not containing $A$ or $Z$. Denote by $k$ the intersection multiplicity of $\Gamma$ with $H$. Then $A$ and $Z$ are separated by $H$ if $k$ is odd and not separated otherwise.*

*Proof.* Connect $Z$ and $A$ by a segment so as to obtain a closed polygon $\overline{\Gamma}$ and consider a hyperplane $H'$ close to $H$, transverse to $\overline{\Gamma}$ and intersecting $\Gamma$ in $k$ points. Since $\overline{\Gamma}$ is contractible, $H'$ intersects $\overline{\Gamma}$ in an even number of points. Therefore, $H'$ intersects the segment $(Z, A)$ for odd $k$ and does not intersect it for even $k$. $\square$

The next definition introduces a significant class of polygons which is our main object of study.

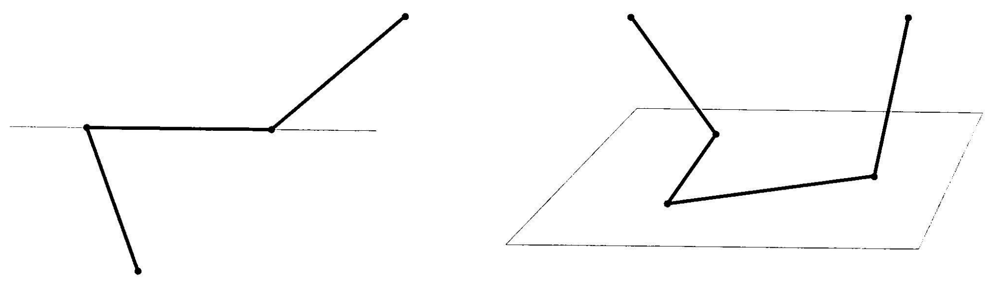
DEFINITION 3.6. A polygon $P$ is called *strictly convex* if through every $d-1$ vertices there passes a hyperplane $H$ whose intersection multiplicity with $P$ is equal to $d-1$.

This definition becomes, in the smooth limit, that of strict convexity for smooth curves, due to Barner.

DEFINITION 3.7. A $d$-tuple of consecutive vertices $(V_i, \dots, V_{i+d-1})$ of a polygon $P$ in $\mathbf{RP}^d$ is called a *flattening* if the endpoints $V_{i-1}$ and $V_{i+d}$ of the broken line $(V_{i-1}, \dots, V_{i+d})$ are:

(a) separated by the hyperplane through $(V_i, \dots, V_{i+d-1})$ if $d$ is even,

(b) not separated if $d$ is odd.



a) $d=2$ b) $d=3$

FIGURE 4

REMARK 3.8. A curve in $\mathbf{RP}^d$ can be lifted to $\mathbf{R}^{d+1} \setminus \{0\}$; the lifting is not unique. Given a polygon $P \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_n$, we lift it to $\mathbf{R}^{d+1}$ as a polygon $\widetilde{P}$ and denote its vertices by $\widetilde{V}_1, \ldots \widetilde{V}_n$. Then a $d$-tuple $(V_i, \ldots, V_{i+d-1})$ is a flattening if and only if the determinant

$$\Delta_j = \left| \widetilde{V}_j \ \ldots \ \widetilde{V}_{j+d} \right| \tag{3.1}$$

changes sign as $j$ varies from $i-1$ to $i$.

This property is independent of the lifting.

### 3.2 A SIMPLEX IS STRICTLY CONVEX

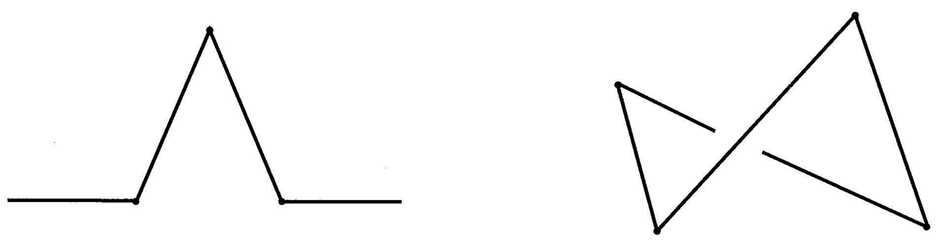
Define a simplex $S_d \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_{d+1}$ as the projection from the punctured $\mathbf{R}^{d+1}$ of the polygonal line:

$$\widetilde{V}_1 = (1,0,\ldots,0), \quad \widetilde{V}_2 = (0,1,0,\ldots,0), \quad \ldots, \widetilde{V}_{d+1} = (0,\ldots,0,1) \tag{3.2}$$

and

$$\widetilde{V}_{d+2} = (-1)^{d+1} \widetilde{V}_1 . \tag{3.3}$$

The last vertex has the same projection as the first one; $S_d$ is contractible for odd $d$, and non-contractible for even $d$.



a) $d = 2$ b) $d = 3$

FIGURE 5

PROPOSITION 3.9. *The polygon $S_d$ is strictly convex.*

*Proof.* We need to prove that through every $(d-1)$-tuple

$$(V_1, \ldots, \widehat{V}_i, \ldots, \widehat{V}_j, \ldots, V_{d+1})$$

there passes a hyperplane $H$ intersecting $P$ with multiplicity $d-1$. Select a point $W$ on the line $(\widetilde{V}_i, \widetilde{V}_j)$ in such a manner that $W$ lies on the segment $(\widetilde{V}_i, \widetilde{V}_j)$ if $j-i$ is even, and does not lie on it if $j-i$ is odd. Define $\widetilde{H}$ as the linear span of $\widetilde{V}_1, \ldots, \widehat{\widetilde{V}}_i, \ldots, \widehat{\widetilde{V}}_j, \ldots, \widetilde{V}_{d+1}, W$. We claim that its projection $H \subset \mathbf{RP}^d$ meets $S_d$ with multiplicity $\leq d-1$.

Let $H'$ be a hyperplane close to $H$ and transverse to $S_d$; assume, further, that $H'$ contains no vertices. It is enough to show that $H'$ cannot intersect $S_d$ in more than $d-1$ points. On the one hand, $H'$ cannot intersect all the edges of $S_d$. Or else, $\widetilde{H'}$ would separate all pairs of consecutive vertices, and this would contradict the choice of $W$. On the other hand, if the number of intersections of $H'$ and $S_d$ were greater than $d-1$, it would be equal to $d+1$. Indeed, for topological reasons, the parity of this intersection number is that of $d-1$. We obtain a contradiction, which proves the claim.

Finally, by Lemma 3.3, the intersection multiplicity of $H$ with $S_d$ is not less than $d-1$. □

A curious property of a simplex is that each of its $d$-tuples of vertices is a flattening.

LEMMA 3.10. *The simplex $S_d$ has $d+1$ flattenings.*

*Proof.* The determinant (3.1) involves all $d+1$ vectors $\widetilde{V}_1, \dots, \widetilde{V}_{d+1}$. If $d$ is odd then, according to (3.3), $\widetilde{V}_{d+2} = \widetilde{V}_1$, and we are reduced to the fact that a cyclic permutation of vectors changes the sign of the determinant. On the other hand, if $d$ is even then $\widetilde{V}_{d+2} = -\widetilde{V}_1$, which also leads to a change of sign in (3.1). □

## 3.3 BARNER'S THEOREM FOR POLYGONS

Now we formulate the result which serves as the main technical tool in the proof of Theorems 2.2, 2.6 and 2.10. Recall that we consider generic polygons in $\mathbf{RP}^d$ with at least $d+1$ vertices.

THEOREM 3.11. *A strictly convex polygon in $\mathbf{RP}^d$ has at least $d+1$ flattenings.*

*Proof.* Induction on the number $n$ of vertices.

Induction starts with $n = d+1$. Up to projective transformations, the unique strictly convex $(d+1)$-gon is the simplex $S_d$. Indeed, every generic $(d+1)$-tuple of points in $\mathbf{RP}^d$ can be taken into any other one by a projective transformation. Therefore, all generic broken lines with $d$ edges are projectively equivalent. It remains for us to connect the last point with the first one, and there are exactly two ways of doing this. One yields a contractible polygon, and the other a non-contractible one. One of these polygons is $S_d$, while the other one cannot be strictly convex, since its intersection number with a hyperplane does not have the same parity as $d-1$. The base for induction is then provided by Lemma 3.10.

Let $P$ be a strictly convex $(n+1)$-gon with vertices $V_1, \dots, V_{n+1}$. Delete $V_{n+1}$ and connect $V_n$ with $V_1$ in such a way that the new edge $(V_n, V_1)$, together with the two deleted ones, $(V_n, V_{n+1})$ and $(V_{n+1}, V_1)$, form a contractible triangle. Denote the new polygon by $P'$.

Let us show that $P'$ is strictly convex. $P$ is strictly convex, therefore through every $d-1$ vertices of $P'$ there passes a hyperplane $H$ intersecting $P$ with multiplicity $d-1$. We want to show that the intersection multiplicity of $H$ with $P'$ is also $d-1$. Let $H'$ be a hyperplane close to $H$ and transverse to $P$ and $P'$. The intersection number of $H'$ with $P'$ does not exceed that with $P$. Indeed, if $H'$ intersects the new edge, then it intersects one of the deleted ones since the triangle is contractible.

By the induction hypothesis, $P'$ has at least $d+1$ flattenings. To prove the theorem, it remains for us to show that $P'$ cannot have more flattenings than $P$.

Consider the sequence of determinants (3.1) $\Delta_1, \Delta_2, \dots, \Delta_{n+1}$. On replacing $P$ by $P'$ we remove $d+1$ consecutive determinants

$$\Delta_{n-d+1},\ \Delta_{n-d+2},\ \dots,\ \Delta_{n+1} \tag{3.4}$$

and replace them with $d$ new determinants

$$\Delta'_{n-d+1},\ \Delta'_{n-d+2},\ \dots,\ \Delta'_n, \tag{3.5}$$

where

$$\Delta'_{n-d+i} = \left|\widetilde{V}_{n-d+i} \ \dots\ \widehat{\widetilde{V}}_{n+1}\ \dots\ \widetilde{V}_{n+i+1}\right| \tag{3.6}$$

with $i = 1, \dots, d$. The transition from (3.4) to (3.5) is done in two steps. Firstly, we add (3.5) to (3.4) so that the two sequences alternate, that is, we put $\Delta'_j$ between $\Delta_j$ and $\Delta_{j+1}$. And secondly, we delete the "old" determinants (3.4). We will prove that the first step preserves the number of sign changes, while the second step obviously cannot increase this number.

LEMMA 3.12. *If $\Delta_{n-d+i}$ and $\Delta_{n-d+i+1}$ have the same sign, then $\Delta'_{n-d+i}$ is also of the same sign.*

*Proof of the lemma.* Since $P$ is in general position, the removed vector $\widetilde{V}_{n+1}$ is a linear combination of $d+1$ vectors $\widetilde{V}_{n-d+i}, \dots, \widetilde{V}_n, \widetilde{V}_{n+2}, \dots, \widetilde{V}_{n+i+1}$:

$$\widetilde{V}_{n+1} = a\widetilde{V}_{n-d+i} + b\widetilde{V}_{n+i+1} + \cdots, \tag{3.7}$$

where the dots indicate a linear combination of the remaining vectors. It follows from (3.6) that

$$\Delta_{n-d+i} = (-1)^{i-1}\, b\, \Delta'_{n-d+i}, \quad \Delta_{n-d+i+1} = (-1)^{d-i}\, a\, \Delta'_{n-d+i}. \tag{3.8}$$

It is time to use the strict convexity of $P$. Let $H$ be a hyperplane in $\mathbf{RP}^d$ through $d-1$ vertices $V_{n-d+i+1}, \dots, \widehat{V}_{n+1}, \dots, V_{n+i}$ which intersects $P$ with multiplicity $d-1$, and let $\widetilde{H}$ be its lifting to $\mathbf{R}^{d+1}$. Choose a linear function $\varphi$ in $\mathbf{R}^{d+1}$ vanishing on $\widetilde{H}$ and such that $\varphi(\widetilde{V}_{n+1}) > 0$. We claim that

$$(-1)^{d-i}\varphi(\widetilde{V}_{n-d+i}) > 0 \quad \text{and} \quad (-1)^{i-1}\varphi(\widetilde{V}_n) > 0. \tag{3.9}$$

Indeed, by Lemma 3.3, the intersection multiplicities of $\widetilde{H}$ with the polygonal lines $(\widetilde{V}_{n-d+i}, \dots, \widetilde{V}_{n+1})$ and $(\widetilde{V}_{n+1}, \dots, \widetilde{V}_{n+i+1})$ are at least $d-i$ and $i-1$, respectively. Since $H$ intersects $P$ with multiplicity $d-1$, the above two multiplicities are indeed equal to $d-i$ and $i-1$. The inequalities (3.9) now readily follow from Lemma 3.5.

Finally, we evaluate $\varphi$ on (3.7):

$$\varphi(\widetilde{V}_{n+1}) = a\,\varphi(\widetilde{V}_{n-d+i}) + b\,\varphi(\widetilde{V}_{n+i+1}).$$

It follows from (3.9) and the inequality $\varphi(\widetilde{V}_{n+1}) > 0$ that at least one of the numbers $(-1)^{i-1}b$ and $(-1)^{d-i}a$ is positive. In view of (3.8), Lemma 3.12 follows. □

Thus Theorem 3.11 is also proved. □

REMARK 3.13. Strict convexity is necessary for the existence of $d+1$ flattenings. One can easily construct a closed polygon without any flattenings and even $C^0$-approximate an arbitrary closed smooth curve by such polygons. In the smooth case such an approximation is well known: given a curve $\gamma_0$, the approximating one, $\gamma$, spirals around in a tubular neighbourhood of $\gamma_0$. In the polygonal case we take a sufficiently fine straightening of $\gamma$.

## 4. APPLICATIONS OF THE MAIN THEOREM

### 4.1 PROOF OF THEOREMS 2.2, 2.6 AND 2.10

Now we prove the results announced in Section 2. The idea is the same in all three cases and is precisely that of Barner's proof of the smooth versions of these theorems – see [3] and also [15]. We will consider Theorem 2.6 in detail, indicating the necessary changes in the other two cases.

Let $P$ be as in Theorem 2.6. We consider the Veronese mapping $\mathcal{V}\colon \mathbf{RP}^2 \to \mathbf{RP}^5$ of degree 2:

$$\mathcal{V}\colon (x:y:z) \mapsto (x^2:y^2:z^2:xy:yz:zx)\,. \tag{4.1}$$

The image $\mathcal{V}(P)$ is a piecewise smooth curve. Every edge is homotopic to a straight segment, with the endpoints $\mathcal{V}(V_i), \mathcal{V}(V_{i+1})$ fixed, and we obtain a polygon $Q$ in $\mathbf{RP}^5$. Assume first that $Q$ is in general position.

LEMMA 4.1. *A quintuple $\mathcal{V}(V_i), \dots, \mathcal{V}(V_{i+4})$ is a flattening of $Q$ if and only if $(V_i, \dots, V_{i+4})$ is an extremal quintuple of vertices of $P$.*

*Proof.* The Veronese map establishes a one-to-one correspondence between conics in $\mathbf{RP}^2$ and hyperplanes in $\mathbf{RP}^5$ : the image of a conic is the intersection of a hyperplane with the quadric surface $\mathcal{V}(\mathbf{RP}^2)$. Since $\mathcal{V}$ is an embedding, the points $V_{i-1}$ and $V_{i+5}$ lie on one side of the conic through $(V_i, \dots, V_{i+4})$ if and only if the points $\mathcal{V}(V_{i-1})$ and $\mathcal{V}(V_{i+5})$ lie on one side of the corresponding hyperplane. $\square$

Next we show that the polygon $Q$ is strictly convex. Given 4 indices $i_1, i_2, i_3, i_4$, we consider two lines in $\mathbf{RP}^2$ : $(V_{i_1}, V_{i_2})$ and $(V_{i_3}, V_{i_4})$; the union of these lines is a conic that does not meet $P$ any more. The corresponding hyperplane in $\mathbf{RP}^5$ contains the vertices $\mathcal{V}(V_{i_1}), \mathcal{V}(V_{i_2}), \mathcal{V}(V_{i_3}), \mathcal{V}(V_{i_4})$ and intersects $Q$ with multiplicity 4.

Theorem 2.6 now follows from Theorem 3.11 for $d = 5$, provided $Q$ is in general position. Otherwise, we replace $P$ by a convex polygon $P'$, close to $P$, such that the corresponding polygon $Q'$ is in general position. Then, as above, $P'$ has at least 6 extremal quintuples of vertices, and therefore so does $P$. This completes the proof.

To prove Theorems 2.2 and 2.10, one replaces the map (4.1) by the Veronese map $\mathcal{V} : \mathbf{RP}^2 \to \mathbf{RP}^3$

$$\mathcal{V} : (x:y:z) \mapsto (x^2+y^2 : z^2 : yz : zx)$$

and by the Segre map $\mathcal{S}\colon \mathbf{RP}^1 \times \mathbf{RP}^1 \to \mathbf{RP}^3$

$$\mathcal{S}\colon ((x_1:y_1),(x_2:y_2)) \mapsto (x_1x_2 : x_1y_2 : y_1x_2 : y_1y_2)\,,$$

respectively. The proofs of strict convexity for the corresponding polygons $Q$ reproduce those in the smooth case (see [15]).

### 4.2 Concluding remarks

It would be interesting to provide discrete analogues of other "4-vertex type" theorems known in the smooth case, and to find their specifically discrete proofs. We give two examples.

The following statement is a discrete version of the celebrated Möbius theorem (in dimension 2, "flattening" means "inflection") – see [9]:

*An embedded non-contractible closed polygon in* $\mathbf{RP}^2$ *has at least* 3 *flattenings.*

The notion of flattening for a polygonal line extends, in an obvious way, from $\mathbf{RP}^d$ to the sphere $S^d$. One has the following statement:

*An embedded closed polygon in* $S^2$ *bisecting the area has at least* 4 *flattenings.*

In the smooth case this was proved by B. Segre [14] and by V. Arnold (see [1, 2]).

We are confident that these statements hold true and can be proved in a similar way as in the smooth case. However, a detailed discussion would go beyond the limits of this article.

In conclusion, let us formulate a conjecture. For $k \geq d+2$ the following statement is stronger than Theorem 3.11.

Conjecture 4.2. *A strictly convex polygon in* $\mathbf{RP}^d$ *that intersects a hyperplane with multiplicity* $k$ *has at least* $k$ *flattenings.*

In the smooth case this is precisely Barner's result in full generality [3]. Conjecture 4.2 would imply strengthenings of Theorems 2.2, 2.6 and 2.10 – see [15] for the smooth case. For instance, the following result would hold.

*Let* $X$ *and* $Y$ *be two* $n$*-tuples of points in* $\mathbf{RP}^1$ *(see Section 2.3). If the closed broken line* $\big((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\big)$ *in* $\mathbf{RP}^1 \times \mathbf{RP}^1$ *intersects the graph of a projective transformation with multiplicity* $k$*, then there exist at least* $k$ *extremal triples of indices.*

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