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THEOREM 2. Let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 2$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (regarded over \mathbf{R}) have infinitely many non-isomorphic rational forms.

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras \mathfrak{g} (two of them appear in Theorem 2 for p = c = 2) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let $G = UT_3(\mathbf{R})$ be the Lie group of upper triangular 3×3 -matrices with 1 on the diagonal, $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$ be Lie algebra of G. Consider $G \times G$ and its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ which has infinitely many non-isomorphic rational forms \mathfrak{h}_m ($m \ge 1$ is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let Γ_m and Γ_n be corresponding lattices in $G \times G$ for distinct m, n. One can prove that the ratio of the covolumes of Γ_m and Γ_n with respect to a Haar measure on $G \times G$ equals $m\sqrt{m}/n\sqrt{n}$ up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3 Γ_m and Γ_n are not commensurable in any sense.

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2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra \mathfrak{g} has a \mathbb{Q} -form \mathfrak{h} and \mathfrak{i} (resp. \mathfrak{a}) is an ideal (resp. a subalgebra) of \mathfrak{g} . We say that \mathfrak{i} (resp. \mathfrak{a}) is rational if $\mathfrak{i} \cap \mathfrak{h}$ (resp. $\mathfrak{a} \cap \mathfrak{h}$) is a rational form of \mathfrak{i} (resp. \mathfrak{a}). For instance, the terms $C^k\mathfrak{g}$ of the lower central series of \mathfrak{g} are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$ is a rational form of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$. Let

(2.1)
$$\mathfrak{g} = \mathfrak{i}_1 > \mathfrak{i}_2 > \cdots > \mathfrak{i}_{k+1} = 0$$

be a descending series of rational ideals of \mathfrak{g} . We say that a basis $X = \{x_1, \ldots, x_d\}$ of a rational form \mathfrak{h} is based on (2.1) if x_1, \ldots, x_{p_1} generate $\mathfrak{g} \mod \mathfrak{i}_2, x_1, \ldots, x_{p_2}$ generate $\mathfrak{g} \mod \mathfrak{i}_3$ and so on. It can be shown that such a basis exists for any series (2.1). In the sequel we will use these kinds of bases for a suitable descending series dealing, for instance, with Heisenberg algebras.

Recall that the (generalized) Heisenberg algebra $\mathfrak{hei}_{2k+1}(\mathbf{R})$ has an **R**-basis H_1, \ldots, H_{2k+1} in which

$$(2.2) [H_1, H_2] = [H_3, H_4] = \cdots = [H_{2k-1}, H_{2k}] = H_{2k+1},$$

other brackets being trivial. Here the 1-dimensional centre is spanned by H_{2k+1} .

Given an extension (1.1) one can attach to it a 2-cocycle $\omega: \Lambda^2 \mathfrak{a} \to \mathbf{R}$ in the usual way. Also ω can be regarded as a symplectic form on \mathfrak{a} . If $\mathfrak{b} = \mathfrak{h}\mathfrak{e}\mathfrak{i}_{2k+1}(\mathbf{R})$ then ω is the canonical non-degenerate symplectic form with respect to the basis $H_1, \ldots, H_{2k} \pmod{\mathbf{R} \cdot H_{2k+1}}$.

Let $d = \dim_{\mathbf{R}} \mathfrak{a}$ and let $m = d - rank(\omega)$ be the codimension of the kernel of ω . It is not hard to see (cf. the proof of the proposition below) that the Lie algebra \mathfrak{b} is uniquely defined up to **R**-isomorphism by d and m. Namely,

$$\mathfrak{b} \cong \mathfrak{hei}_{d+1-m}(\mathbf{R}) \oplus \mathbf{R}^m$$
.

This implies that the centre of \mathfrak{b} is (m + 1)-dimensional. Thus, two Lie algebras \mathfrak{b}_1 and \mathfrak{b}_2 $(\dim_{\mathbf{R}} \mathfrak{b}_1 = \dim_{\mathbf{R}} \mathfrak{b}_2)$ of type (1.1) are not isomorphic if $m_1 \neq m_2$.

Evidently, $\mathfrak{hei}_{2k+1}(\mathbf{Q})$ is a rational form for $\mathfrak{hei}_{2k+1}(\mathbf{R})$.

The following proposition holds.

PROPOSITION 2.1. In the above notation let \mathfrak{h} be a rational form of \mathfrak{b} . Let $d = \dim(\mathfrak{b}) - 1$, $m = \dim[\mathfrak{b}, \mathfrak{b}]$ and let \mathbf{Q}^m denote the abelian Lie \mathbf{Q} -algebra of dimension m. Then

$$\mathfrak{h} \cong \mathfrak{hei}_{d+1-m}(\mathbf{Q}) \oplus \mathbf{Q}^m$$

over \mathbf{Q} , i.e., there is a unique rational form for \mathfrak{b} up to isomorphism.

Proof. Choose a **Q**-basis B_1, \ldots, B_{d+1} for \mathfrak{h} . Either all brackets $[B_i, B_j] = 0$, and then $\mathfrak{h} \cong \mathbf{Q}^{d+1}$, or there are i, j such that $[B_i, B_j] = C \neq 0$.

We may suppose that $C = B_{d+1}$. Thus the derived subalgebra of \mathfrak{h} is spanned by B_{d+1} . The corresponding symplectic form ω is represented by a

skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis B_1, \ldots, B_d (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij}B_{d+1}$. Over **Q** one can choose a canonical symplectic basis $\widehat{B_1}, \ldots, \widehat{B_d}$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix \widehat{M} representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to 2l and 2l = d - m. In the basis B_1, \ldots, B_{d+1} (we omit the 'hats') of \mathfrak{h}

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 EXAMPLE OF A FREE NILPOTENT ALGEBRA

Let $f_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $f_c(n, \mathbf{R})$ has a unique rational form $f_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \ldots, x_n, \ldots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that x_1, \ldots, x_n span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, \mathfrak{h} is generated by $\{x_1, \ldots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi: \mathfrak{f}_c(n, \mathbf{Q}) \to \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of \mathfrak{h} equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 MORE EXAMPLES

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let \mathfrak{g}_t , $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \ldots, x_6\}$ such that

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = tx_5, \quad [x_1, x_5] = x_6, [x_2, x_3] = x_4, \quad [x_2, x_4] = x_5, \quad [x_3, x_4] = x_6,$$

other brackets being trivial. One can show that

- 1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \ldots, x_6 \rangle$, $k = 2, \ldots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of \mathfrak{g}_t .
- 2. The centralizer \mathfrak{C} of $C^4\mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4\mathfrak{g}_t] = 0\}$ is spanned by x_2, \ldots, x_6 .

- 3. Real Lie algebras \mathfrak{g}_0 and \mathfrak{g}_1 are not isomorphic but $\forall t \neq 0$ $\mathfrak{g}_t \cong \mathfrak{g}_1$.
- 4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over \mathbf{Q} .
- 5. g_0 and g_1 are two Lie algebras with a unique rational form up to isomorphism.
- 6. Let \mathfrak{g} be a split real simple Lie algebra of type G_2 , \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of \mathfrak{g} with respect to \mathfrak{h} . Then \mathfrak{n}_+ is isomorphic to \mathfrak{g}_0 .

3. MALCEV'S EXAMPLE

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a **Q**-isomorphism between \mathfrak{g}_t and \mathfrak{g}_s . It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$ and the centralizer \mathfrak{c} of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by x_3, \ldots, x_6 .

$$y_{1} = a_{11}x_{1} + a_{12}x_{2} + a_{13}x_{3} + a_{14}x_{4} + \dots$$

$$y_{2} = a_{21}x_{1} + a_{22}x_{2} + a_{23}x_{3} + a_{24}x_{4} + \dots$$

$$y_{3} = a_{33}x_{3} + a_{34}x_{4} + \dots$$

$$y_{4} = a_{44}x_{4} + \dots$$

We do not explicit the expressions for y_5, y_6 . Here y_1, \ldots, y_6 are basis elements of \mathfrak{g}_s satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1,y_2]=y_4=\Delta x_4+\ldots,$$

 $\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

(3.1)
$$\begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases}$$

We need to compute the remaining two brackets. First of all,

$$(3.2) \quad [y_1, y_3] = a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] = a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 = (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6.$$

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