4. Concentration property and fixed points

EXAMPLE 11. Let π be a strongly continuous unitary representation of a compact group G in ℓ_2. Then ℓ_2 decomposes into the orthogonal direct sum of finite-dimensional (irreducible) unitary G-modules, $\ell_2 \cong \bigoplus_{n=1}^{\infty} V_n$. Set for each $n \in \mathbf{N}$

$$\mathbf{S}_n = \mathbf{S}^{\infty} \cap \bigoplus_{i=1}^{n} V_n .$$

We obtain a nested sequence of spheres of increasing finite dimension which are invariant under the action of G. Let μ_n denote the rotation-invariant probability measure on the sphere $\mathbf{S}_n$. Denote also $G_n = G$ for all n. Then $(G, \mathbf{S}^{\infty})$ is a Lévy transformation group.

4. CONCENTRATION PROPERTY AND FIXED POINTS

The following definition is an attempt to capture 'concentration in the absence of measure' (as indeed there are typically no invariant measures on infinite dimensional spaces).

DEFINITION 7 [M2,M3]. Let a group G act on a metric space X by uniform isomorphisms. Call a subset $A \subseteq X$ *essential* if for every $\varepsilon > 0$ and every finite collection $g_1, \ldots, g_N \in G$ one has

$$\bigcap_{i=1}^{N} g_i A_{\varepsilon} \neq \varnothing .$$

(Have another look at Fig. 1 !)

EXERCISE 5. The definition obtained by replacing $g_i A_{\varepsilon}$ with $(g_i A)_{\varepsilon}$ is equivalent.

Informally speaking, an essential set is so 'big' that translates of any ε-neighbourhood of it, taken in any finite number, don't fit in without overlapping.

DEFINITION 8 (*ibidem*). A G-space X has the *concentration property* if every finite cover of X contains at least one essential set.

Perhaps one gets a better idea of the property if we start with an example where it is violated.

EXAMPLE 12 (Imre Leader, 1988, unpublished). The $U(\mathcal{H})$-space $\mathbf{S}^\infty$ (the unit sphere in $\mathcal{H} = \ell_2$) does not have the concentration property. Denote by E the set of all even natural numbers, and let P_E be the corresponding projection in ℓ_2. Set

$$A = \{x \in \mathbf{S}^\infty : \|P_E x\| \geq \sqrt{2}/2\},$$
$$B = \{x \in \mathbf{S}^\infty : \|P_E x\| \leq \sqrt{2}/2\}.$$

Clearly, $A \cup B = \mathbf{S}^\infty$. At the same time, both A and B are inessential. Indeed, let E_1, E_2, E_3 be three arbitrary disjoint infinite subsets of $\mathbf{N}$, and let $\varphi_i : \mathbf{N} \to \mathbf{N}$ be bijections with $\varphi_i(E) = E_i$, $i = 1, 2, 3$. Let g_i denote the unitary operator on $\ell_2(\mathbf{N})$ induced by φ_i. Now

$$g_i(A) = \{x \in \mathbf{S}^\infty : \|P_{E_i} x\| \geq \sqrt{2}/2\},$$

and consequently

$$(g_i(A))_\varepsilon \subseteq \{x \in \mathbf{S}^\infty : \|P_{E_i} x\| \geq (\sqrt{2}/2) - \varepsilon\}.$$

Thus, as long as $\varepsilon < \sqrt{2}/2 - \sqrt{3}/3$, we have

$$\bigcap_{i=1}^{3} (g_i(A))_\varepsilon = \varnothing.$$

The set B is treated similarly.

THEOREM 2. *A compact G-space K has the concentration property if and only if it contains a fixed point: $g \cdot \kappa = \kappa$ for all $g \in G$.*

Proof. ($\Rightarrow$) CLAIM 1. *There is a point $\kappa \in K$ such that every neighbourhood of κ is essential.*

Assuming the contrary, we could have covered K with inessential open sets and, selecting a finite open subcover, obtain a contradiction.

CLAIM 2. *Any point κ as above is G-fixed.*

Again, assume that for some $g \in G$, $g \cdot \kappa \neq \kappa$. Set $\varepsilon = d(\kappa, g \cdot \kappa)/2$. Choose a number $\delta > 0$ so small that $\delta \leq \varepsilon/2$ and the g-translate of the open ball $B_\delta(\kappa)$ is contained in the $(\varepsilon/2)$-ball around $g \cdot \kappa$. The set $V = B_\delta(\kappa)$ is essential, yet the δ-neighbourhoods of V and $g \cdot V$ don't meet, a contradiction.

($\Leftarrow$) Obvious. □

The following result provides nontrivial examples of G-spaces with the concentration property.

THEOREM 3. *Every Lévy G-space (G,X) has the concentration property.*

Proof. Let
$$\gamma = \{A_1, A_2, \dots, A_k\}$$
be a finite cover of X. Since for each $n = 1,2,\dots$ the values $\mu_n(A_i)$, $i = 1,2,\dots,k$, add up to one, at least one of the sets in γ, let us denote it simply $A = A_i$, has the property:
$$\limsup_{n\to\infty} \mu_n(A) \geq \frac{1}{k}\,.$$
Now let $\varepsilon > 0$ and a finite collection $g_j, j = 1,2,\dots,m$ be given. Using Exercise 2, choose a number n_0 so large that
$$\mu_n(B_\varepsilon) > 1 - \frac{1}{m}$$
whenever $n > n_0$ and $\mu_n(B) \geq \frac{1}{k}$. Choose an $n > n_0$ with $\mu_n(A) \geq \frac{1}{k}$; then $\mu_n(g_jA) \geq \frac{1}{k}$ as well, and
$$\mu_n(g_jA)_\varepsilon > 1 - \frac{1}{m}, \quad i = 1,2,\dots,m\,,$$
implying that the ε-neighbourhoods of all the translates of A_ε by g_j's have a common point. □

To extract useful information from the above, it only remains to link the concentration property of a G-space to that of its compactification.

LEMMA 1. *Let X and Y be two G-spaces*[3]*. Let $i\colon X \to Y$ be an equivariant map. If (G,X) has the concentration property, then so does (G,Y).*

Proof. If $A \subseteq X$ is an essential subset, then so is $i(A)$. Notice that the uniform continuity of i is used here in a substantial way. □

The following is now immediate.

THEOREM 4 [Gr-M1]. *Let (G,X) be a Lévy G-space and let K be a compact G-space, such that there is an equivariant map $X \to K$. Then K has a G-fixed point.* □

---

[3]) As before, X and Y are metric spaces upon which G acts continuously, by uniform isomorphisms.

Using Theorem 1 and Example 10, we obtain

COROLLARY 1. *Whenever the topological group* $\mathrm{U}(\ell_2)_s$ *acts continuously on a compact space, it has a fixed point.*

Such topological groups are said to have the *fixed point on compacta property,* or else to be *extremely amenable.* And indeed, this property is a drastically strengthened form of the usual amenability, which can be reformulated as follows (Day): a topological group G is amenable if and only if every affine continuous action of G on a convex compact set [in a locally convex space] has a fixed point.

REMARK 5. No locally compact group can have the fixed point on compacta property, this is a theorem by Veech ([Ve], Th. 2.2.1).

REMARK 6. The unitary group $\mathrm{U}(\mathcal{H})_s$ was the first 'natural' extremely amenable group to be discovered. The second such discovery was the group $L_0((0,1),\mathbf{T})$ of all (equivalence classes of) measurable maps from the unit interval to the circle rotation group, equipped with the topology of convergence in measure. This was proved by Glasner (and published years later [Gl]) and, independently, by Furstenberg and Weiss (never published). This group is a Lévy group, and the approximating Lévy family of subgroups is formed by tori $\mathbf{T}^n$, made up of simple functions with respect to a refining sequence of measurable partitions of $(0,1)$.

It is interesting that both groups mentioned in the previous paragraph appear as the 'outermost' cases of a newly discovered class of extremely amenable groups. Recall that a von Neumann algebra M is *approximately finite-dimensional* if it contains a directed family of finite-dimensional $*$-subalgebras with everywhere dense union. Denote by M_* the predual of M. It is proved in [G-P] that a von Neumann algebra M is approximately finite-dimensional if and only if the unitary group of M, equipped with the topology $\sigma(M,M_*)$, is the product of a compact group with an extremely amenable group.

The two cases to consider now are $M=\mathcal{B}(\mathcal{H})$, where the unitary group with the above topology is $\mathrm{U}(\mathcal{H})_s$, and $M=L^\infty(0,1)$, in which case the unitary group is $L_0((0,1),\mathbf{T})$.

As a corollary, nuclear C^*-algebras admit a characterization in terms of topological dynamics of their unitary groups. Recall that an action of a group G on a compact space X is *minimal* if the G-orbit of every point of X is everywhere dense, and *equicontinuous* if the family of all mappings

$x \mapsto gx$, $g \in G$ of X to itself is uniformly equicontinuous. By considering the enveloping von Neumann algebra, one can deduce that a C^*-algebra A is nuclear if and only if every minimal continuous action of the unitary group $\mathrm{U}(A)$, equipped with the $\sigma(A, A^*)$-topology, on a compact space K is equicontinuous.

REMARK 7. One has to be careful while applying Theorem 4. For instance, consider the infinite permutation group S_∞, formed by all self-bijections of a countably infinite set, say $\mathbf{Z}$. This group is equipped with the natural Polish topology of pointwise convergence on discrete $\mathbf{Z}$, induced by the embedding $S_\infty \hookrightarrow \mathbf{Z}^{\mathbf{Z}}$. The idea of applying concentration in finite groups of permutations (Example 3) to conclude that S_∞ is a Lévy group is attractive, but does not work.

EXERCISE 6. Let d be any right-invariant metric on S_∞, generating the topology of pointwise convergence. Show that S_∞, acting on the left upon (S_∞, d), does not have the concentration property.

[*Hint.* Let τ be the transposition exchanging 0 and 1 and leaving the rest of $\mathbf{Z}$ fixed. Choose $\varepsilon > 0$ so that the ε-ball around e_G is contained in the intersection of the isotropy subgroups of 0 and 1. Now partition S_∞ into two sets A and B, where

$$A = \{\sigma \in S_\infty \colon \sigma^{-1}(0) < \sigma^{-1}(1)\}$$

and $B = S_\infty \setminus A$. Try to apply the concentration property to the cover $\{A, B\}$, the number ε, and the collection of two elements e, τ.]

It follows that S_∞ acts on some compact space without fixed points. (This was noted in [P1].) Very recently such an action was constructed explicitely by Eli Glasner and Benji Weiss [Gl-W]. We will return to their construction later (Subsection 6.4).

One can even show that S_∞ is not a Lévy group no matter what the group topology is ([P2], Remark 4.9). However, it is still possible to put the finite permutation groups (S_n) together so as to obtain a Lévy group.

This is the group $\mathrm{Aut}(X, \mu)$ of all measure-preserving automorphisms of the standard non-atomic Lebesgue space, (X, μ), equipped with the weak topology, that is, the weakest topology making every map of the form $\mathrm{Aut}(X, \mu) \ni \tau \mapsto \mu(A \,\Delta\, \tau(A)) \in \mathbf{R}$ continuous, where $A \subseteq X$ is a measurable set. This group contains finite permutation groups, realized as subgroups of interval exchange transformations, and any right-invariant metric makes those subgroups into a Lévy family. A similar result holds for the group $\mathrm{Aut}^*(X,\mu)$ of all measure class preserving transformations (Thierry Giordano and the author [G-P]).

5. Invariant means on spheres

Let a group G act on a metric space X by uniform isomorphisms. The formula

$$^{g}f(x) = f(g^{-1}\cdot x)$$

determines an action of G on the space $\mathrm{UCB}(X)$ of all uniformly continuous bounded complex valued functions on X by linear isometries. If G is a topological group acting on X continuously, the above action of G on $\mathrm{UCB}(X)$ need not, in general, be continuous. (An example: $G = \mathrm{U}(\ell_2)_s$, $X = \mathbf{S}^\infty$.) However, the action will be continuous if X is compact. (An easy check.) To some extent, the latter observation can be inverted.

Exercise 7. Let a topological group G act continuously on a commutative unital C^*-algebra A by automorphisms. Then this action determines a continuous action of G on the space of maximal ideals of A, equipped with the usual (weak*) topology.

Recall that a *mean* on a space $\mathcal{F}$ of functions is a positive linear functional, m, of norm one, sending the function 1 to 1. A mean is *multiplicative* if $\mathcal{F}$ is an algebra and the mean is a homomorphism of this algebra to $\mathbf{C}$.

Corollary 2. *Let (G,X) be a Lévy G-space. Then there exists a G-invariant multiplicative mean on the space* $\mathrm{UCB}(X)$ *of all bounded uniformly continuous functions on X.*

Proof. According to Exercise 7, the group G acts continuously on the space $\mathfrak{M}$ of maximal ideals of the C^*-algebra $\mathrm{UCB}(X)$. Therefore, $\mathfrak{M}$ is an equivariant compactification of X. By Theorem 4, there is a fixed point $\varphi\in\mathfrak{M}$, which is the desired invariant multiplicative mean. □

The following is deduced by considering Example 11.