TOPOLOGICAL PROOF OF THE GROTHENDIECK FORMULA IN REAL ALGEBRAIC GEOMETRY

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A TOPOLOGICAL PROOF OF THE GROTHENDIECK FORMULA IN REAL ALGEBRAIC GEOMETRY

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INTRODUCTION

In 1973 A. Tognoli [28], improving upon earlier work of J. Nash [25], demonstrated that every closed (compact without boundary) smooth manifold M is diffeomorphic to a nonsingular algebraic subset X of $\mathbf{R}^n$ for some n. The reader can also consult [11, Theorem 14.1.10] for a proof that requires the reading of only a few pages of [11]. This remarkable result of Nash-Tognoli soon gave rise to a larger program. By carefully choosing X one wanted to realize algebraically not only M alone, but also some objects such as submanifolds, vector bundles, homology or cohomology classes, etc. attached to it.

Examples of successes include the relative Nash-Tognoli theorem dealing with finite collections of smooth submanifolds of M [1] and a theorem asserting that X can be selected in such a way that every topological real vector bundle on X is isomorphic to an algebraic vector bundle [9]. A special case of the relative Nash-Tognoli theorem was used in [2] to obtain an elegant topological characterization of real algebraic sets with isolated singularities. A conjecture was put forward that X can be chosen with each homology class in $H_*(X; \mathbf{Z}/2)$ represented by an algebraic subset of X [3]; this would have simplified many constructions and facilitated a topological characterization of all real algebraic sets.

However, the conjecture was refuted in [7] by the following argument. For any integer $m \geq 11$, there exist an m-dimensional closed smooth manifold M

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and a cohomology class v in $H^2(M;\mathbf{Z}/2)$ such that v cannot be represented as the second Stiefel-Whitney class of a real vector bundle on M (it is now known that $m \geq 11$ can be replaced by $m \geq 6$, which is sharp [27]). On the other hand, for any compact nonsingular real algebraic set X, each cohomology class in $H^2(X;\mathbf{Z}/2)$, whose Poincaré dual homology class can be represented by an algebraic subset of X, is the second Stiefel-Whitney class of some algebraic vector bundle on X. Therefore the conjecture has to be false.

We call the latter part of the argument the *Grothendieck formula in real algebraic geometry*. This was proved in [7] in two steps. First a proof of the Grothendieck formula relating vector bundles and algebraic cycles on schemes over $\mathbf{R}$ was sketched (an analog of the formula from earlier papers [18, 19] for varieties over an algebraically closed field); this sketch contains some flaws. Then a connection, established in [15], between the Chern classes with values in the Chow ring and the Stiefel-Whitney classes yielded the conclusion. The appearance of [17] allowed for a shorter proof [12], based on the same principles and free from the flaws mentioned above. According to the authors' experience such proofs still present considerable difficulty for many topologically inclined mathematicians. The goal of this paper is to give a self-contained topological proof that uses only the simplest facts from algebra. Several applications of the Grothendieck formula in real algebraic geometry, besides the one discussed above, are contained in [12, 13, 22].

The paper assumes knowledge of singular homology and cohomology with coefficients in $\mathbf{Z}/2$ at the level of [26]. Real vector bundles and their Stiefel-Whitney classes, for which a good reference is [24], are also used. All smooth (of class $\mathcal{C}^\infty$) manifolds are assumed to be paracompact and without boundary. From real algebraic geometry we require only a few notions, recalled here and elucidated in detail in just a few pages of [5], [8], or [11]. Basic and generally well-known facts from commutative algebra that are needed can all be found in [23].

1. THE GROTHENDIECK FORMULA

REAL ALGEBRAIC VARIETIES

The Zariski topology on $\mathbf{R}^n$ is the topology for which the closed sets are precisely the algebraic subsets of $\mathbf{R}^n$. Let V be a nonempty Zariski locally closed subset of $\mathbf{R}^n$ (that is, V is the difference of two algebraic subsets of $\mathbf{R}^n$). The *dimension* $\dim V$ of V is the largest integer d for which there exist an open subset N of $\mathbf{R}^n$ (in the usual metric topology) and polynomials $P_1, \dots, P_{n-d}$ in $\mathbf{R}[T_1, \dots, T_n]$ such that $N \cap V$ is a nonempty set, $V \subset Z$, $N \cap V = N \cap Z$, where

$$Z = \{z \in \mathbf{R}^n \mid P_1(z) = \dots = P_{n-d}(z) = 0\},$$

and the Jacobian matrix $\left[\frac{\partial P_i}{\partial T_j}(z)\right]$, $1 \le i \le n-d$, $1 \le j \le n$, has rank $n-d$ for every point z in $N \cap V$ (several other characterizations of $\dim V$ are given in [8, Sect. 3.4 and 11, Sect. 2.8]). A point x in V is said to be *nonsingular* if one can find N and $P_1, \dots, P_{n-d}$ as above, with x in $N \cap V$ and $d = \dim V$; otherwise x is called singular (this agrees with [5, 11], whereas in [8] a slightly different definition is used, with the condition $d = \dim V$ omitted). Clearly, the set of all nonsingular points of V is a smooth submanifold of $\mathbf{R}^n$ of dimension $\dim V$. Consider V endowed with the Zariski topology induced from $\mathbf{R}^n$. The set $\operatorname{Sing}(V)$ of all singular points of V is Zariski closed in V and

$$\dim \operatorname{Sing}(V) < \dim V$$

[5, p. 28 or 8, p. 137 or 11, p. 69]. If $\operatorname{Sing}(V)$ is empty, V is said to be nonsingular.

Recall that V is irreducible if it cannot be represented as the union of two Zariski closed subsets of V, distinct from V. Assuming that V is irreducible, one has $\dim W < \dim V$ for every Zariski closed subset W of V, $W \ne V$ [5, p. 28 or 8, p. 136 or 11, p. 50]. If V is not irreducible, then $V = V_1 \cup \dots \cup V_k$, where $V_1, \dots, V_k$ are irreducible Zariski closed subsets of V, with V_i not contained in V_j for $i \ne j$; the sets $V_1, \dots, V_k$ are uniquely determined and called the irreducible components of V [5, p. 20 or 8, p. 119 or 11, p. 50].

A function $f \colon V \to \mathbf{R}$ is said to be *regular* if for every point x in V, there exist an open neighborhood (in the Zariski topology) U_x of x in V and two polynomials P and Q in $\mathbf{R}[T_1, \dots, T_n]$ such that $Q(y) \ne 0$ and $f(y) = P(y)/Q(y)$ for all y in U_x. In fact, one can take $U_x = V$, and hence f is always a quotient of two polynomials $f = P/Q$ with $Q(y) \ne 0$ for all y in V [5, p. 19 or 8, p. 121 or 11, p. 62]. The set $\mathcal{R}(V)$ of all regular functions on V forms a ring under pointwise addition and multiplication.

Throughout this paper, a *real algebraic variety* is, by definition, a Zariski locally closed subset of $\mathbf{R}^n$, for some n. A map $\varphi \colon V \to W$ between real algebraic varieties, $W \subset \mathbf{R}^p$, is called *regular* if each component φ_i of $\varphi = (\varphi_1, \dots, \varphi_p)$ is in $\mathcal{R}(V)$. If, moreover, φ is bijective and φ^{-1} is regular, we call φ *biregular*. One easily sees that nonsingular points and dimension are invariant under biregular maps [5, p. 28, or 8, p. 126 or 11, p. 67].

Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the usual metric topology.

COMBINATORIAL PROPERTIES OF REAL ALGEBRAIC VARIETIES

Recall that the semialgebraic subsets of $\mathbf{R}^n$ form the smallest family of subsets containing all sets of the form

$$\{x \in \mathbf{R}^n \mid P(x) > 0\}, \text{ where } P \text{ is in } \mathbf{R}[T_1, \dots, T_n],$$

and closed under taking finite unions, finite intersections, and complements. Obviously, any algebraic subset of $\mathbf{R}^n$ is semialgebraic.

We shall make use of the following important result (for its proof cf. [8, Theorem 2.6.12] or [11, Theorem 9.2.1]):

THEOREM 1.1. *Let T be a compact semialgebraic set. Given a finite family $\mathcal{F}$ of semialgebraic subsets of T, there exists a semialgebraic triangulation of T compatible with $\mathcal{F}$.*

In other words, there exist a simplicial complex K and a homeomorphism $\Phi\colon |K| \to T$, where $|K|$ is the polyhedron determined by K, such that for each open simplex σ of K and each set S in $\mathcal{F}$, the image $\Phi(\sigma)$ is a semialgebraic subset of T, which is either contained in or disjoint from S.

For any pair (X, A) of topological spaces, the Euler-Poincaré characteristic $\chi(X, A)$ is defined by

$$\chi(X, A) = \sum_{r \geq 0} (-1)^r \dim_{\mathbf{Z}/2} H_r(X, A; \mathbf{Z}/2),$$

provided that $\dim_{\mathbf{Z}/2} H_r(X, A; \mathbf{Z}/2)$ is finite for all $r \geq 0$ and equals 0 for all r large enough (if the homology group $H_*(X, A; \mathbf{Z})$ is finitely generated, then this defintion coincides with the usual one [16, Proposition VI.7.21]). If $\chi(X)$ and $\chi(A)$ are defined, then $\chi(X, A)$ is also defined and $\chi(X, A) = \chi(X) - \chi(A)$ [16, Proposition V.5.7]. If K is a finite simplicial complex and ℓ_r is the number of r-simplices in K, then

$$\chi(|K|) = \sum_{r \geq 0} (-1)^r \ell_r .$$

Note that for any compact real algebraic variety V, the Euler-Poincaré characteristic $\chi(V, V \setminus \{x\})$ is defined for every point x of V. Indeed, by Theorem 1.1, there exists a triangulation $\Phi\colon |K| \to V$ of V such that $\Phi(v) = x$ for some vertex v of K. If L is the subcomplex of K of all simplices that do not have v as a vertex, then $|L|$ is a deformation retract of $|K| \setminus \{v\}$, and hence

$$\chi(V, V\setminus\{x\}) = \chi(|K|, |K|\setminus\{v\}) = \chi(|K|, |L|) = \chi(|K|) - \chi(|L|).$$

It follows that

$$\chi(V, V\setminus\{x\}) = \sum_{r\geq 0}(-1)^r m_r\,, \tag{1.2}$$

where m_r is the number of r-simplices of K having v as a vertex.

THEOREM 1.3. *Let V be a compact real algebraic variety. Then for every point x in V, the Euler-Poincaré characteristic $\chi(V, V\setminus\{x\})$ is an odd integer.*

Reference for the proof. It is proved in [8, Theorem 3.10.4], by a nice topological argument, that

$$\sum_{r\geq 0} \dim_{\mathbf{Z}/2} H_r(V, V\setminus\{x\}; \mathbf{Z}/2)$$

is an odd integer. This is equivalent to Theorem 1.3. $\square$

COROLLARY 1.4. *Let V be a compact d-dimensional real algebraic variety and let $\Phi\colon |K| \to V$ be a triangulation of V. Then for any $(d-1)$-simplex σ of K, the number $n(\sigma)$ of d-simplices of K having σ as a face is even.*

Proof. Let $\tau_1, \dots, \tau_{n(\sigma)}$ be the d-simplices of K having σ as a face. Let K' be the barycentric subdivision of K and let b be the barycenter of σ. Denote by n_i the number of simplices s of the barycentric subdivision of τ_i such that b is a vertex of s and s is not in the barycentric subdivision of σ. One readily sees that n_i is odd. Let n be the number of simplices in the barycentric subdivision of σ having b as a vertex. Clearly, n is odd. Note that $n + n_1 + \cdots + n_{n(\sigma)}$ is the number of simplices of K' having b as a vertex. In view of (1.2) and Theorem 1.3, $n(\sigma)$ has to be even. Hence the proof is complete. $\square$

ALGEBRAIC CYCLES

Given a compact d-dimensional real algebraic variety V, we shall now define a homology class $[V]$ in $H_d(V, \mathbf{Z}/2)$ playing a special role in various problems concerning geometry and topology of varieties.

Choose a semialgebraic triangulation of V (Theorem 1.1). By Corollary 1.4, the sum of all d-simplices of this triangulation is a cycle with coefficients in $\mathbf{Z}/2$. The homology class $[V]$ in $H_d(V,\mathbf{Z}/2)$ represented by this cycle is independent of the choice of the triangulation. Indeed, taking any two semialgebraic triangulations of V we can, using Theorem 1.1, find a third one, which is a common subdivision of the two. The uniqueness of $[V]$ follows immediately.

The excision property implies that for each nonsingular point x of V, the image of $[V]$ by the canonical homomorphism

$$H_d(V;\mathbf{Z}/2) \longrightarrow H_d(V, V\setminus\{x\};\mathbf{Z}/2) \cong \mathbf{Z}/2$$

is nonzero. The class $[V]$ is called the *fundamental class* of V. If V is nonsingular, then $[V]$ coincides with the fundamental class of V regarded as a manifold. For other, equivalent, definitions of the fundamental class, cf. [10, 14, 15].

Let X be a compact real algebraic variety. For any d-dimensional Zariski closed subset V of X, we call the element $[V]_X = i_*([V])$ of $H_d(X;\mathbf{Z}/2)$, where $i\colon V \hookrightarrow X$ is the inclusion map, the *homology class of X represented by V*. Denote by

$$H_d^{\mathrm{alg}}(X;\mathbf{Z}/2)$$

the subgroup of $H_d(X;\mathbf{Z}/2)$ generated by all homology classes of X represented by d-dimensional Zariski closed subsets of X. Given two d-dimensional Zariski closed subsets V_1 and V_2 of X, we have $[V_1]_X+[V_2]_X = [W]_X$, where W is the union of the irreducible d-dimensional components of $V_1 \cup V_2$ not contained in $V_1 \cap V_2$. It follows that every element of $H_d^{\mathrm{alg}}(X;\mathbf{Z}/2)$ is of the form $[V]_X$ for some d-dimensional Zariski closed subset V of X.

Assuming that X is compact and nonsingular, we set

$$H^c_{\mathrm{alg}}(X;\mathbf{Z}/2) = D_X^{-1}(H_d^{\mathrm{alg}}(X;\mathbf{Z}/2)),$$

where $c+d = \dim X$ and $D_X\colon H^c(X;\mathbf{Z}/2) \to H_d(X;\mathbf{Z}/2)$ is the Poincaré duality isomorphism, $D_X(u) = u\cap[X]$ for every u in $H^c(X;\mathbf{Z}/2)$. The groups $H_d^{\mathrm{alg}}(X;\mathbf{Z}/2)$ and $H^c_{\mathrm{alg}}(X;\mathbf{Z}/2)$ are important invariants of compact nonsingular real algebraic varieties. The reader can refer to [14] for a short survey of their properties and applications, and for a more extensive list of references. These groups have the expected functorial properties, which however will neither be proved nor used here.

THE GROTHENDIECK FORMULA

In order to state the Grothendieck formula, we have to recall the definition of an algebraic vector bundle.

An *algebraic vector bundle* on a real algebraic variety X is a triple $\xi = (E, \pi, X)$, where E is a real algebraic variety, $\pi\colon E \to X$ is a regular map, and the following conditions are satisfied:

(i) for every point x in X, the fiber $E_x = \pi^{-1}(x)$ is a real vector space,

(ii) there exist a finite cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X by Zariski open sets, and for each λ in Λ, a nonnegative integer k and a biregular map $\varphi\colon \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbf{R}^k$ such that $\varphi(E_x) = \{x\} \times \mathbf{R}^k$ and the restriction $E_x \to \{x\} \times \mathbf{R}^k$ of φ is a linear isomorphism for every x in U_λ,

(iii) ξ is an algebraic subbundle of the trivial vector bundle $X \times \mathbf{R}^p$, for some p.

Condition (iii) means that there exists a regular map $i\colon E \to X \times \mathbf{R}^p$ such that $i(E_x) \subseteq \{x\} \times \mathbf{R}^p$ and the restriction $E_x \to \{x\} \times \mathbf{R}^p$ of i is an injective linear map for every x in X.

Basic properties of algebraic vector bundles can be found in [11, Chapter 12]. The reader should keep in mind that algebraic vector bundles considered here are sometimes called strongly algebraic vector bundles in the literature [9, 10, 13].

Our main goal is to give a self-contained proof of the following Grothendieck formula.

THEOREM 1.5. *Let X be a compact nonsingular real algebraic variety. For every cohomology class v in $H^2_{\mathrm{alg}}(X;\mathbf{Z}/2)$, there exists an algebraic vector bundle ξ on X with $w_1(\xi) = 0$ and $w_2(\xi) = v$.*

Here $w_k(-)$ stands for the k^{th} Stiefel-Whitney class.

We end this section by stating two results whose proofs use, in an essential way, the Grothendieck formula.

Given a compact smooth manifold M, let us denote by $\mathrm{Vect}(M)$ the set of isomorphism classes of topological real vector bundles on M and define

$$W^2(M) = \{v \in H^2(M;\mathbf{Z}/2) \mid v = w_2(\xi) \quad \text{for some } \xi \text{ in } \mathrm{Vect}(M)\}.$$

One easily sees that $W^2(M)$ is a subgroup of $H^2(M;\mathbf{Z}/2)$. As mentioned in the introduction, in general, $W^2(M) \neq H^2(M;\mathbf{Z}/2)$ for $\dim M \geq 6$. The group $W^2(M)$ plays a crucial role in the problem of representation of homology classes in codimension 2 by Zariski closed subsets. More precisely, we have the following result.

THEOREM 1.6. *Let M be a compact orientable smooth manifold of dimension at least* 5 *and let G be a subgroup of $H^2(M;\mathbf{Z}/2)$. Then the following conditions are equivalent:*

(*a*) *There exist a nonsingular real algebraic variety X and a diffeomorphism $\varphi\colon X \to M$ such that $\varphi^*(G) = H^2_{\mathrm{alg}}(X;\mathbf{Z}/2)$.*

(*b*) *$w_2(M) \in G \subseteq W^2(M)$, where $w_2(M)$ is the second Stiefel-Whitney class of M.*

Proof. See [13].

Another application concerns the problem of approximation of smooth curves (that is, one-dimensional smooth submanifolds) by algebraic curves. First recall that a compact smooth submanifold N of a nonsingular real algebraic variety X is said to *admit an algebraic approximation* in X if for each neighborhood $\mathcal{U}$ of the inclusion map $N \hookrightarrow X$ (in the $\mathcal{C}^\infty$ topology on the set $\mathcal{C}^\infty(N,X)$ of smooth maps from N into X), there exists a smooth embedding $e\colon N \to X$ such that e is in $\mathcal{U}$ and $e(N)$ is a nonsingular Zariski closed subset of X.

THEOREM 1.7. *Let X be a compact nonsingular real algebraic variety of dimension* 3 *and let C be a compact smooth curve in X. Then C admits an algebraic approximation in X if and only if the $\mathbf{Z}/2$-homology class represented by C is in $H_1^{\mathrm{alg}}(X;\mathbf{Z}/2)$.*

The proof of Theorem 1.7 will be given elsewhere. Under the extra assumption that C is connected and homologous to the union of finitely many nonsingular real algebraic curves in X the theorem is proved in [4].

2. PROOF OF THE GROTHENDIECK FORMULA

We shall use homology and cohomology groups with coefficients exclusively in $\mathbf{Z}/2$ and therefore we shall suppress the coefficient group in our notation.

For any continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces, we let

$$f_*\colon H_k(X,A) \to H_k(Y,B)\,, \quad f^*\colon H^k(Y,B) \to H^k(X,A)$$

denote the induced homomorphisms.

For the convenience of the reader we shall now review some facts from topology. Let B be a paracompact topological space and let $\xi = (E, \pi, B)$ be a real vector bundle of rank k on B. Let $s_0\colon B \to E$ be the zero section of ξ, that is, $s_0(x) = 0_x$ for all x in B, where 0_x is the zero vector in the fiber $E_x = \pi^{-1}(x)$. We set $0_E = s_0(B)$. Recall that the Thom class τ_ξ of ξ is a unique element of $H^k(E, E\smallsetminus 0_E)$ such that for every point x in B, the homomorphism

$$H^k(E, E\smallsetminus 0_E) \to H^k(E_x, E_x\smallsetminus\{0_x\}) \cong \mathbf{Z}/2\,,$$

induced by the inclusion map $(E_x, E_x\smallsetminus\{0_x\}) \hookrightarrow (E, E\smallsetminus 0_E)$, sends τ_ξ to the generator of $\mathbf{Z}/2$ [24, Theorem 8.1] (the name "Thom class" is not used in [24]). For every nonnegative integer q, we have the Thom isomorphism

$$\varphi_q\colon H^q(B) \to H^{k+q}(E, E\smallsetminus 0_E)$$
$$\varphi_q(v) = \pi^*(v) \cup \tau_\xi \quad \text{for all } v \text{ in } H^q(B)$$

[24, Definition 8.2].

If $s\colon B \to E$ is any continuous section of ξ and $\bar{s}\colon (B, B\smallsetminus s^{-1}(0_E)) \to (E, E\smallsetminus 0_E)$ is the map defined by s, then

$$w_k(\xi) = i^*(\bar{s}^*(\tau_\xi))\,, \tag{2.1}$$

where $i\colon B = (B, \varnothing) \hookrightarrow (B, B\smallsetminus s^{-1}(0_E))$ is the inclusion map. Indeed, let $j\colon E \hookrightarrow (E, E\smallsetminus 0_E)$ be the inclusion map. Note that $H\colon E \times [0,1] \to (E, E\smallsetminus 0_E)$, defined by $H(e,t) = (1-t)j(e) + t(\bar{s}\circ i\circ\pi)(e)$ for all (e,t) in $E\times[0,1]$, is a homotopy between j and $\bar{s}\circ i\circ \pi$. In particular, $j^* = (\bar{s}\circ i\circ\pi)^* = \pi^*\circ i^*\circ \bar{s}^*$, and hence

$$\pi^*(i^*(\bar{s}^*(\tau_\xi))) \cup \tau_\xi = j^*(\tau_\xi) \cup \tau_\xi = \tau_\xi \cup \tau_\xi\,,$$

where the last equality is the standard property of the cup product [26, p. 251, property 8]. Thus $\varphi_k(i^*(\bar{s}^*(\tau_\xi))) = \tau_\xi \cup \tau_\xi$. Now, (2.1) follows since $w_k(\xi) = \varphi_k^{-1}(\tau_\xi \cup \tau_\xi)$ [24, p. 91].

Let M be a smooth m-dimensional manifold and let N be a smooth n-dimensional submanifold of M. Assume that N is a closed subset of M. A tubular neighborhood of N in M is a smooth real vector bundle $\xi = (E, \pi, N)$ on N such that E is an open neighborhood of N in M and $0_E = N$ [20]. By the excision property, the inclusion map $e\colon (E, E\setminus N) \hookrightarrow (M, M\setminus N)$ induces an isomorphism

$$e^*: H^k(M, M\setminus N) \to H^k(E, E\setminus N),$$

where $k = m - n$. The Thom class τ_N^M of N in M is a unique element of $H^k(M, M\setminus N)$ such that $e^*(\tau_N^M) = \tau_\xi$. The Thom isomorphism yields

$$H^k(M, M\setminus N) \cong H^0(N).$$

Hence

$$\tau_N^M \quad \text{generates} \quad H^k(M, M\setminus N) \cong \mathbf{Z}/2, \tag{2.2}$$

provided N is connected. Assuming that N has exactly r connected components $N_1, \dots, N_r$, the inclusion maps $e_i\colon (M, M\setminus N) \hookrightarrow (M, M\setminus N_i)$ give rise to an isomorphism

$$t\colon \bigoplus_{i=1}^{r} H^k(M, M\setminus N_i) \to H^k(M, M\setminus N)$$

$$t(u_1, \dots, u_r) = e_1^*(u_1) + \cdots + e_r^*(u_r)$$

satisfying

$$t(\tau_{N_1}^M, \dots, \tau_{N_r}^M) = \tau_N^M. \tag{2.3}$$

If $f\colon M \to P$ is a smooth map between smooth manifolds, transverse to a smooth submanifold Q of P (Q a closed subset of P) and with $N = f^{-1}(Q)$, then

$$\bar{f}^*(\tau_Q^P) = \tau_N^M, \tag{2.4}$$

where $\bar{f}\colon (M, M\setminus N) \to (P, P\setminus Q)$ is the map defined by f. Indeed, after a homotopy, f looks like a vector bundle map between tubular neighborhoods of N and Q [20, p. 117, Theorem 6.7], and hence (2.4) follows from the definition of the Thom class.

Let Δ be the diagonal of $M \times M$,

$$\Delta = \{(x, y) \in M \times M \mid x = y\},$$

and let τ in $H^m(M \times M, (M \times M)\setminus\Delta)$ be the Thom class of Δ in $M \times M$. For every point x in M, the image of τ under the homomorphism

$$H^m(M \times M, (M \times M)\setminus\Delta) \to H^m(M, M\setminus\{x\}) \cong \mathbf{Z}/2$$

induced by the map $(M, M\setminus\{x\}) \to (M\times M, (M\times M)\setminus\Delta)$, $y \to (x, y)$, generates $\mathbf{Z}/2$ [24, Lemma 11.7]. Thus τ is the orientation class of M over $\mathbf{Z}/2$ in the terminology used in [26, p. 294]. For any pair (A,B) of subsets of M, $B \subseteq A$, and any integer q satisfying $0 \leq q \leq m$, let

$$\gamma_{A,B} \colon H_q(A,B) \to H^{m-q}(M \setminus B,\, M \setminus A)$$

be the homomorphism defined by

$$\gamma_{A,B}(a) = a \diagdown j^*_{A,B}(\tau)\,,$$

where $\diagdown$ is the slant product and

$$j_{A,B} \colon (A \times (M \setminus B), (A \times (M \setminus A)) \cup (B \times (M \setminus B))) \hookrightarrow (M \times M, (M \times M) \setminus \Delta)$$

is the inclusion map, cf. [26, p. 351]. If B is empty, we shall write γ_A instead of $\gamma_{A,\varnothing}$. The following naturality property is satisfied: if (A',B') is another pair of subsets of M, $B' \subseteq A'$, and $A \subseteq A'$, $B \subseteq B'$, then the diagram

$$\text{(2.5)} \qquad \begin{array}{ccc} H_q(A,B) & \xrightarrow{\gamma_{A,B}} & H^{m-q}(M \setminus B, M \setminus A) \\ \downarrow & & \downarrow \\ H_q(A',B') & \xrightarrow{\gamma_{A',B'}} & H^{m-q}(M \setminus B', M \setminus A')\,, \end{array}$$

where the vertical homomorphisms are induced by the appropriate inclusion maps, is commutative [26, pp. 287, 289, 351]. Furthermore, if M is compact, then

$$\text{(2.6)} \qquad \gamma_M = D_M^{-1}\,,$$

that is,

$$\gamma_M \colon H_q(M) \to H^{m-q}(M)$$

is the inverse of the Poincaré duality isomorphism

$$D_M \colon H^{m-q}(M) \to H_q(M)\,, \quad D_M(u) = u \cap [M]\,.$$

This follows from [26, p. 305, Theorem 12] and the fact that, in the notation of [26, p. 353, Lemma 15], θ is the identity map, provided $X = Y$, $G = \mathbf{Z}/2$.

We shall also make use of the following result.

PROPOSITION 2.7. *If M is compact and (A,B) is a compact polyhedral pair in M, then*

$$\gamma_{A,B} \colon H_q(A,B) \to H^{m-q}(M \setminus B,\, M \setminus A)$$

is an isomorphism.

Proof. We have the following diagram:

$$\begin{array}{ccc}
H_q(B) & \xrightarrow{\gamma_B} & H^{m-q}(M, M\setminus B) \\
\downarrow & & \downarrow \\
H_q(A) & \xrightarrow{\gamma_A} & H^{m-q}(M, M\setminus A) \\
\downarrow & & \downarrow \\
H_q(A,B) & \xrightarrow{\gamma_{A,B}} & H^{m-q}(M\setminus B, M\setminus A) \\
\downarrow & & \downarrow \\
H_{q-1}(B) & \xrightarrow{\gamma_B} & H^{m-q+1}(M, M\setminus B) \\
\downarrow & & \downarrow \\
H_{q-1}(A) & \xrightarrow{\gamma_A} & H^{m-q+1}(M, M\setminus A),
\end{array}$$

where the columns are parts of the long exact sequences for the pair (A, B) and the triple $(M, M\setminus B, M\setminus A)$. By (2.5) and [26, p. 287, property 3, and p. 351], the diagram is commutative. It is proved in [26, p. 351, Lemma 14] that γ_A and γ_B are isomorphisms for q and $q-1$. In view of the five lemma, $\gamma_{A,B}$ is also an isomorphism. $\square$

After this preparation, we are ready to prove an auxiliary result relating homology and cohomology of real algebraic varieties. Let X be a compact n-dimensional nonsingular real algebraic variety and let V be a d-dimensional Zariski closed subset of X. By Theorem 1.1, V is a compact polyhedron and hence

$$\gamma_V : H_d(V) \to H^c(X, X\setminus V),$$

where $c = n - d$, is an isomorphism in view of Proposition 2.7. For our purposes it is important to give a characterization of $\gamma_V([V])$. Set $S = \mathrm{Sing}(V)$ and let

$$i: (X\setminus S, (X\setminus S)\setminus(V\setminus S)) \hookrightarrow (X, X\setminus V),\ j: X \hookrightarrow (X, X\setminus V)$$

be the inclusion maps (of course, $X\setminus V = (X\setminus S)\setminus(V\setminus S)$). Since $V\setminus S$ is a d-dimensional nonsingular Zariski closed subset of $X\setminus S$, the Thom class $\tau_{V\setminus S}^{X\setminus S}$ in $H^c(X\setminus S, (X\setminus S)\setminus(V\setminus S))$ is defined.

PROPOSITION 2.8. *There exists a unique element τ_V^X in $H^c(X, X \setminus V)$ such that*

$$i^*(\tau_V^X) = \tau_{V \setminus S}^{X \setminus S}.$$

Furthermore,

$$\tau_V^X = \gamma_V([V]) \quad \textit{and} \quad D_X(j^*(\tau_V^X)) = [V]_X.$$

Proof. We shall first prove $i^*(\gamma_V([V])) = \tau_{V \setminus S}^{X \setminus S}$. The smooth manifold $V \setminus S$ is a semialgebraic set and therefore has finitely many connected components, say $N_1, \dots, N_r$ [11, p. 35]. If V_i is the closure of N_i in V and $S_i = V_i \cap S$, then $N_i = V_i \setminus S_i$. Note that V_i and S_i are compact semialgebraic subsets of V [8, p. 61 or 11, p. 27]. By (2.5), we have the following commutative diagram:

$$\begin{array}{ccccc}
H_d(V) & \xrightarrow{\varphi} & H_d(V,S) & \xleftarrow{\alpha} & \bigoplus_{i=1}^{r} H_d(V_i, S_i) \\
\gamma_V \downarrow & & \gamma_{V,S} \downarrow & & \bigoplus_{i=1}^{r} \gamma_{V_i,S_i} \downarrow \\
H^c(X, X \setminus V) & \xrightarrow{i^*} & H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S)) & \xleftarrow{\beta} & \bigoplus_{i=1}^{r} H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i),
\end{array}$$

where φ is induced by the appropriate inclusion map, whereas

$$\alpha(a_1, \dots, a_r) = \alpha_1(a_1) + \cdots + \alpha_r(a_r),$$
$$\beta(u_1, \dots, u_r) = \beta_1(u_1) + \cdots + \beta_r(u_r),$$

with

$$\alpha_i \colon H_d(V_i, S_i) \to H_d(V, S)$$
$$\beta_i \colon H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \to H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S))$$

induced by the inclusion maps.

Since $N_1, \dots, N_r$ are the connected components of the smooth manifold $V \setminus S$, we have another commutative diagram:

$$\begin{array}{ccc}
H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S)) & \xleftarrow{\beta} & \bigoplus_{i=1}^{r} H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \\
t \uparrow & & \bigoplus_{i=1}^{r} \psi_i \downarrow \\
\bigoplus_{i=1}^{r} H^c(X \setminus S, (X \setminus S) \setminus N_i) & \xleftarrow{id} & \bigoplus_{i=1}^{r} H^c(X \setminus S, (X \setminus S) \setminus N_i),
\end{array}$$

where

$$\psi_i \colon H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \to H^c(X \setminus S, (X \setminus S) \setminus N_i)$$

is the homomorphism induced by the appropriate inclusion map and t is the isomorphism of (2.3). It follows from the definiton of the Thom class that

$$(a) \qquad \psi_i(\tau_{N_i}^{X\setminus S_i}) = \tau_{N_i}^{X\setminus S}.$$

Hence, in view of (2.2), ψ_i is an isomorphism of cyclic groups isomorphic to $\mathbf{Z}/2$. Applying (2.3) and (a), we get

$$(b) \qquad \beta(\tau_{N_1}^{X\setminus S_1}, \dots, \tau_{N_r}^{X\setminus S_r}) = \tau_{V\setminus S}^{X\setminus S}.$$

Since, by Proposition 2.7, γ_{V_i,S_i} is an isomorphism, the group $H_d(V_i, S_i)$ is isomorphic to $\mathbf{Z}/2$; let a_i be its unique generator. Now, (a) and (b) imply

$$\gamma_{V,S}(\alpha(a_1, \dots, a_r)) = \tau_{V\setminus S}^{X\setminus S}.$$

Thus in order to verify $i^*(\gamma_V([V])) = \tau_{V\setminus S}^{X\setminus S}$ it suffices to prove

$$(c) \qquad \alpha(a_1, \dots, a_r) = \varphi([V]),$$

which can be done as follows.

Let $\Phi\colon |K| \to V$ be a semialgebraic triangulation of V compatible with $\{V_1, \dots, V_r, S_1, \dots, S_r\}$ (Theorem 1.1). Denote by c_i the chain which is the sum of all d-simplices of K whose images under Φ are contained in V_i. Since $N_i = V_i \setminus S_i$ is a smooth d-dimensional manifold, it follows that every open $(d-1)$-simplex σ of K with $\Phi(\sigma)$ contained in N_i is a face of exactly two d-simplices of K. Thus c_i represents a nonzero homology class in $H_d(V_i, S_i) \cong \mathbf{Z}/2$; in other words, c_i represents a_i. On the other hand, $c_1 + \dots + c_r$ is the sum of all d-simplices of K and therefore it is a cycle representing the fundamental class $[V]$ in $H_d(V)$. Hence (c) follows and $i^*(\gamma_V([V])) = \tau_{V\setminus S}^{X\setminus S}$ is proved.

Let us observe that i^* is injective. Indeed, there is an exact sequence

$$\cdots \to H^c(X, X\setminus S) \to H^c(X, X\setminus V) \to H^c(X\setminus S, X\setminus V) \to \cdots$$

corresponding to the triple $(X, X\setminus S, X\setminus V)$. By Proposition 2.7, $\gamma_S\colon H_d(S) \to H^c(X, X\setminus S)$ is an isomorphism. Since $\dim S < d$, we obtain $H_d(S) = 0$, which implies $H^c(X, X\setminus S) = 0$. Hence i^* is injective as asserted.

Thus $\tau_V^X = \gamma_V([V])$ is a unique element of $H^c(X, X\setminus V)$ satisfying $i^*(\tau_V^X) = \tau_{V\setminus S}^{X\setminus S}$.

It remains to prove $D_X(j^*(\tau_V^X)) = [V]_X$. By (2.5), we have the following commutative diagram:

$$\begin{array}{ccc} H_d(V) & \xrightarrow{e_*} & H_d(X) \\ {\scriptstyle\gamma_V}\downarrow & & \downarrow{\scriptstyle\gamma_X} \\ H^c(X, X\setminus V) & \xrightarrow{j^*} & H^c(X), \end{array}$$

where $e\colon V \hookrightarrow X$ is the inclusion map. In view of (2.6), γ_X is the inverse of D_X and we obtain $D_X(j^*(\tau_V^X)) = e_*([V]) = [V]_X$. Thus the proof is complete. $\square$

We shall now recall a purely algebraic result. Definitions of algebraic terms not explained here can all be found in [23]. Given a ring R (commutative with identity), we let $K_0(R)$ denote the Grothendieck group of finitely generated projective R-modules. If S is a multiplicatively closed subset of R and $S^{-1}R$ denotes the ring of fractions of R with denominators in S, then the canonical ring homomorphism $j_S\colon R \to S^{-1}R$, $j_S(r) = r/1$, induces a group homomorphism $K_0(R) \to K_0(S^{-1}R)$. Assuming that R is a regular ring of finite Krull dimension, every finitely generated R-module has a finite projective resolution [23, p. 208]. The last fact allows one to apply [6, p. 453, Proposition 2.1, p. 492, Proposition 6.1], which yields the result we require: the homomorphism $K_0(R) \to K_0(S^{-1}R)$ is surjective, provided that R is a regular ring of finite Krull dimension (this also easily follows from [23, p. 210, Exercise 4]).

To make use of this result we need some algebraic properties of the ring $\mathcal{R}(X)$ of regular functions on a real algebraic variety X. Suppose that X is a Zariski locally closed subset of $\mathbf{R}^n$ and let $\mathcal{P}(X)$ be the ring of polynomial functions from X into $\mathbf{R}$ ($f\colon X \to \mathbf{R}$ is a polynomial function if for some polynomial P in $\mathbf{R}[T_1, \dots, T_n]$, one has $f(x) = P(x)$ for all x in X). Clearly, $\mathcal{P}(X)$ is a finitely generated $\mathbf{R}$-algebra and thus a Noetherian ring [23, p. 11]. Furthermore, the Krull dimension of $\mathcal{P}(X)$ is equal to $\dim X$ [11, p. 50]. Recall that $\mathcal{R}(X)$ consists of all functions of the form f/g, where f, g are in $\mathcal{P}(X)$ and $g^{-1}(0) = \varnothing$. In other words, $\mathcal{R}(X)$ is the ring of fractions of $\mathcal{P}(X)$ with denominators in the set $\{g \in \mathcal{P}(X) \mid g^{-1}(0) = \varnothing\}$. It follows that $\mathcal{R}(X)$ is a Noetherian ring of Krull dimension $\dim X$ [23, p. 81]. Obviously, for every point x in X,

$$m_x = \{f \in \mathcal{R}(X) \mid f(x) = 0\}$$

is a maximal ideal of $\mathcal{R}(X)$ and each maximal ideal of $\mathcal{R}(X)$ is equal to m_x for some x. The localization $\mathcal{R}(X)_x$ of $\mathcal{R}(X)$ with respect to m_x is a Noetherian local ring of Krull dimension not exceeding $\dim X$ [23, p. 81]. A point x in X is nonsingular if and only if the local ring $\mathcal{R}(X)_x$ is regular of Krull dimension $\dim X$ [11, p. 67]. In particular, the ring $\mathcal{R}(X)$ is regular of finite Krull dimension, provided X is nonsingular. Given a Zariski open subset U of X, the subset

$$S(U) = \{g \in \mathcal{R}(X) \mid g^{-1}(0) \subseteq X \setminus U\}$$

of $\mathcal{R}(X)$ is multiplicatively closed. Since $\mathcal{R}(U) = S(U)^{-1}\mathcal{R}(X)$, it follows from the facts reviewed above that the group homomorphism

$$K_0(\mathcal{R}(X)) \to K_0(\mathcal{R}(U)), \tag{2.9}$$

induced by the restriction ring homomorphism $\mathcal{R}(X) \to \mathcal{R}(U)$, $f \to f|_U$, is surjective, assuming X is nonsingular.

PROPOSITION 2.10. *Let X be a nonsingular real algebraic variety and let U be a Zariski open subset of X. For any algebraic vector bundle η on U, there exists an algebraic vector bundle ξ on X such that $\xi|_U$ and η are algebraically stably equivalent (that is, one can find algebraically trivial vector bundles ϵ_1 and ϵ_2 on U with the property that the bundles $(\xi|_U) \oplus \epsilon_1$ and $\eta \oplus \epsilon_2$ on U are algebraically isomorphic).*

Proof. Let Y be a real algebraic variety. For any algebraic vector bundle ζ on Y, let $\Gamma(\zeta)$ denote the $\mathcal{R}(Y)$-module of algebraic global sections of ζ. One readily proves that the correspondence $\zeta \to \Gamma(\zeta)$ establishes an equivalence of the category of algebraic vector bundles on Y with the category of finitely generated projective $\mathcal{R}(Y)$-modules [11, Proposition 12.1.12]. The proposition follows since (2.9) is surjective. □

Let Y be a real algebraic variety and let W be a Zariski closed subset of Y. Denote by $I_Y(W)$ the ideal of $\mathcal{R}(Y)$ consisting of all regular functions vanishing on W,

$$I_Y(W) = \{f \in \mathcal{R}(Y) \mid f(y) = 0 \text{ for all } y \text{ in } W\}.$$

The restriction homomorphism $\mathcal{R}(Y) \to \mathcal{R}(W)$, $f \to f|_W$, gives rise, for each point y in W, to a ring epimorphism $\mathcal{R}(Y)_y \to \mathcal{R}(W)_y$, whose kernel is equal to the ideal $I_Y(W)\mathcal{R}(Y)_y$ of $\mathcal{R}(Y)_y$. In particular, the quotient ring $\mathcal{R}(Y)_y/I_Y(W)\mathcal{R}(Y)_y$ is isomorphic to $\mathcal{R}(W)_y$. Therefore if y in W is a nonsingular point of Y and $k = \dim Y - \dim W$, then given elements $f_1, \dots, f_k$ of $I_Y(W)$, the following conditions are equivalent:

(i) $I_Y(W)\mathcal{R}(Y)_y = (f_1, \dots, f_k)\mathcal{R}(Y)_y$ and y is a nonsingular point of W,

(ii) $I_Y(W)\mathcal{R}(Y)_y = (f_1, \dots, f_k)\mathcal{R}(Y)_y$ and there exist elements $f_{k+1}, \dots, f_{k+d}$ of $\mathcal{R}(Y)$, $d = \dim W$, such that $f_1, \dots, f_{k+d}$ generate the unique maximal ideal of the local ring $\mathcal{R}(Y)_y$,

(iii) the map $(f_1, \dots, f_k)\colon Y \setminus \mathrm{Sing}(Y) \to \mathbf{R}^k$ is transverse to 0 at y and $W \cap H = f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0) \cap H$, where H is a Zariski open neighborhood of y in $Y \setminus \mathrm{Sing}(Y)$.

Indeed, the equivalence of (i) and (ii) is a consequence of [23, p. 169, Proposition 1.10]. Furthermore, $f_1, \dots, f_{k+d}$ generate the maximal ideal of $\mathcal{R}(Y)_y$ if and only if there exists a neighborhood N of y in $Y \setminus \mathrm{Sing}(Y)$ such that the restriction of $(f_1, \dots, f_{k+d})$ to N is a local coordinate system for the smooth manifold $Y \setminus \mathrm{Sing}(Y)$ [11, pp. 66, 67]. Hence the equivalence of (ii) and (iii) easily follows.

It also follows from [23, p. 169, Proposition 1.10] that $I_Y(W)\mathcal{R}(Y)_y$ is generated by k elements, provided y in W is a nonsingular point of Y and of W.

We shall freely use the facts just reviewed.

Proof of Theorem 1.5. By assumption, $D_X(v) = [V]_X$, where V is a Zariski closed subset of X with $\dim X - \dim V = 2$. If $V_1, \dots, V_p$ are the irreducible components of V of dimension $\dim V$, then $[V]_X = [V_1]_X + \cdots + [V_p]_X$, and hence it suffices to prove the theorem assuming that V is irreducible.

Let x_0 be a nonsingular point of V. Then the ideal $I_X(V)\mathcal{R}(X)_{x_0}$ of the ring $\mathcal{R}(X)_{x_0}$ can be generated by two elements; we choose generators a_1, a_2 that belong to $I_X(V)$. Hence there exists a Zariski open neighborhood U of x_0 in X such that the ideal $I_X(V)\mathcal{R}(U)$ of the ring $\mathcal{R}(U)$ is generated by a_1 and a_2. This implies

$$(a) \qquad I_X(V)\mathcal{R}(U)_x = (a_1, a_2)\mathcal{R}(U)_x \text{ for all } x \text{ in } U\,.$$

Since $\mathrm{Sing}(V)$ is Zariski closed in V, shrinking U if necessary, we may assume that $U \cap \mathrm{Sing}(V) = \varnothing$. Hence from (a), we obtain

$$(b) \quad \text{the map } (a_1, a_2)\colon U \to \mathbf{R}^2 \text{ is transverse to } 0 \text{ in } \mathbf{R}^2 \text{ at each point } x \text{ in } U \cap V\,.$$

Setting $S = V \setminus (U \cap V)$, we have $\mathrm{Sing}(V) \subseteq S$ and, by virtue of irreducibility of V,

$$(c) \qquad \dim S < \dim V\,.$$

Let $Y = X \setminus S$ and $W = V \setminus S$. Then Y is a Zariski open subset of X and W is a Zariski closed subset of Y, with $\dim Y - \dim W = 2$.

CLAIM. *There exist an algebraic vector bundle $\eta = (E, \pi, Y)$ on Y and an algebraic section $s\colon Y \to E$ of η such that η is of rank 2, $W = s^{-1}(0_E)$, and s is transverse to 0_E.*

We prove the claim as follows. Choose a regular function b in $\mathcal{R}(Y)$ with $b^{-1}(0) = W$. Set $b_k = a_k|_Y$ for $k = 1, 2$, and define a map $F\colon Y \times \mathbf{R}^2 \to \mathbf{R}^2$

by

$$F(y,t) = F_t(y) = (b_1(y) + t_1 b(y)^2,\ b_2(y) + t_2 b(y)^2)$$

for all y in Y and $t = (t_1, t_2)$ in $\mathbf{R}^2$.

We assert that F is transverse to 0 in $\mathbf{R}^2$. Indeed, suppose $F(y,t) = 0$ for some (y,t) in $Y \times \mathbf{R}^2$. If y is not in W, then the assertion holds since it suffices to consider the partial derivatives with respect to t_1 and t_2. If y is in W, then (b) implies that $F_t\colon Y \to \mathbf{R}^2$ is transverse to 0 in $\mathbf{R}^2$ at y, which means that the assertion also holds in this case. Hence the assertion is proved.

It follows from the assertion and a standard transversality theorem [20, p. 79, Theorem 2.7] that there exists a point t in $\mathbf{R}^2$ for which the map

$$F_t = (f_1, f_2)\colon Y \to \mathbf{R}^2$$

is transverse to 0 in $\mathbf{R}^2$. Since f_1 and f_2 are in $I_Y(W)$ and W is nonsingular, we get

$$I_Y(W)\mathcal{R}(Y)_y = (f_1, f_2)\mathcal{R}(Y)_y$$

for all y in W. Hence for each point y in W, one can find a Zariski open neighborhood G_y of y in Y with

$$I_Y(W)\mathcal{R}(G_y) = (f_1, f_2)\mathcal{R}(G_y).$$

In particular, $W \cap G_y = f_1^{-1}(0) \cap f_2^{-1}(0) \cap G_y$. Taking G to be the union of the G_y for y in W, we get $W = f_1^{-1}(0) \cap f_2^{-1}(0) \cap G$, which implies

$$(d) \qquad f_1^{-1}(0) \cap f_2^{-1}(0) = W \cup W',$$

where W' is a subset of Y disjoint from W. Clearly, W' is contained in $Y \setminus G$. Since $W \cup W'$ and $Y \setminus G$ are Zariski closed subsets of Y, and $W' = (W \cup W') \cap (Y \setminus G)$, it follows that W' is also Zariski closed in Y. The transversality of $(f_1, f_2)\colon Y \to \mathbf{R}^2$ to 0 in $\mathbf{R}^2$ together with (d) imply

$$(e) \qquad I_Y(W \cup W')\mathcal{R}(Y)_y = (f_1, f_2)\mathcal{R}(Y)_y \text{ for all } y \text{ in } Y.$$

Choosing regular functions ψ_1 and ψ_2 in $\mathcal{R}(Y)$ with $\psi_1^{-1}(0) = W$ and $\psi_2^{-1}(0) = W'$ (this is possible since W and W' are Zariski closed in Y), we see that $\psi_1\psi_2$ belongs to $I_Y(W \cup W')$ and hence

$$\psi_1\psi_2 = h_1 f_1 + h_2 f_2$$

for some regular functions h_1 and h_2 in $\mathcal{R}(Y)$ (the last assertion can easily be deduced directly from (e), but, anyhow, it is also a consequence of (e) and [23, p. 93, Rule 1.1]).

Let $\mathbf{M}_2(\mathbf{R})$ denote the set of all real 2×2 matrices (identified with $\mathbf{R}^4$ and regarded as a real algebraic variety). Consider regular maps $g_{21}\colon U_1 = Y \setminus W \to \mathbf{M}_2(\mathbf{R})$ and $g_{12}\colon U_2 = Y \setminus W' \to \mathbf{M}_2(\mathbf{R})$ defined by

$$g_{21} = \begin{bmatrix} f_1\psi_2/\psi_1 & -h_2/\psi_1^2 \\ f_2\psi_2/\psi_1 & h_1/\psi_1^2 \end{bmatrix}, \quad g_{12} = \begin{bmatrix} h_1/\psi_2^2 & h_2/\psi_2^2 \\ -f_2\psi_1/\psi_2 & f_1\psi_1/\psi_2 \end{bmatrix}.$$

For each point y in $U_1 \cap U_2$, the matrices $g_{12}(y)$ and $g_{21}(y)$ are invertible and $g_{12}(y)g_{21}(y)$ is the identity matrix. Define

$$E = \{(y, v_1, v_2) \in Y \times \mathbf{R}^2 \times \mathbf{R}^2 \mid v_1 = g_{12}(y)\, v_2 \text{ if } y \in U_2 \\ \text{and } v_2 = g_{21}(y)\, v_1 \text{ if } y \in U_1\}$$

and $\pi\colon E \to Y$, $\pi(y, v_1, v_2) = y$. Since $\{U_1, U_2\}$ is a Zariski open cover of Y, it follows that E is a Zariski closed subset of $Y \times \mathbf{R}^2 \times \mathbf{R}^2$. Clearly, π is a regular map and, for each point y in Y, the fiber $E_y = \pi^{-1}(y)$ is a vector subspace of $\{y\} \times \mathbf{R}^2 \times \mathbf{R}^2$. Furthermore, the map

$$U_k \times \mathbf{R}^2 \to \pi^{-1}(U_k),\ (y, v) \to (y, g_{1k}(y) \cdot v,\ g_{2k}(y) \cdot v)$$

is biregular for $k = 1, 2$, where $g_{kk}(y)$ is the identity matrix. Thus $\eta = (E, \pi, Y)$ is an algebraic vector bundle of rank 2 on Y. The map $s\colon Y \to E$

$$s(y) = (y, (\psi_1(y), 0),\ (f_1(y)\psi_2(y),\ f_2(y)\psi_2(y)))$$

is an algebraic section of η with $s^{-1}(0_E) = W$. On U_2 the section s is represented by $(f_1, f_2)\colon U_2 \to \mathbf{R}^2$, and therefore s is transverse to 0_E. Hence the claim is proved.

Let $\bar{s}\colon (Y, Y \setminus W) \to (E, E \setminus 0_E)$ be the map defined by s and let $\ell\colon Y \hookrightarrow (Y, Y \setminus W)$ be the inclusion map. In view of (2.1), we have $w_2(\eta) = \ell^*(\bar{s}^*(\tau_\eta))$, while (2.4) yields $\bar{s}^*(\tau_\eta) = \tau_W^Y$. It follows that

$$(f) \qquad w_2(\eta) = \ell^*(\tau_W^Y).$$

If $i\colon (Y, Y \setminus W) \hookrightarrow (X, X \setminus V)$, $j\colon X \hookrightarrow (X, X \setminus V)$, and $e\colon Y \hookrightarrow X$ are the inclusion maps, then the diagram

$$\begin{tikzcd} H^2(X, X \setminus V) \arrow[r, "i^*"] \arrow[d, "j^*"'] & H^2(Y, Y \setminus W) \arrow[d, "\ell^*"] \\ H^2(X) \arrow[r, "e^*"] & H^2(Y) \end{tikzcd}$$

is commutative.

Since $W \subseteq V \setminus \mathrm{Sing}(V)$, Proposition 2.8 yields

$$(g) \qquad i^*(\tau_V^X) = \tau_W^Y, \qquad j^*(\tau_V^X) = v.$$

By combining (*d*) and (*e*), we get

$$(h) \qquad w_2(\eta) = \ell^*(i^*(\tau_V^X)) = e^*(j^*(\tau_V^X)) = e^*(v).$$

Proposition 2.10 implies that there exists an algebraic vector bundle ζ on X, whose restriction to Y is algebraically stably equivalent to η. In particular, $w_2(\eta) = w_2(\zeta \mid Y) = e^*(w_2(\zeta))$, and hence applying (*h*), we get

$$(i) \qquad e^*(v) = e^*(w_2(\zeta)).$$

Note that e^* is injective. Indeed, there is an exact sequence

$$H^2(X, Y) \longrightarrow H^2(X) \xrightarrow{e^*} H^2(Y).$$

Since $S = X \setminus Y$ is Zariski closed in X, by Theorem 1.1 and Proposition 2.7, $H^2(X, Y)$ is isomorphic to $H_{n-2}(S)$, where $n = \dim X$. Observing that $\dim V = n - 2$ and applying (*c*), we obtain $H_{n-2}(S) = 0$. Thus e^* is injective and (*i*) implies

$$(j) \qquad w_2(\zeta) = v.$$

The vector bundle ζ, being algebraic, has a constant rank on each irreducible component of X. It follows that there exists an algebraic vector bundle ϵ on X such that the restriction of ϵ to each irreducible component of X is algebraically trivial and $\zeta \oplus \epsilon$ has constant rank, say, r on X. The line bundle $\lambda = \Lambda^r(\zeta \oplus \epsilon)$ is algebraic [11, Proposition 12.1.8] and hence the vector bundle $\xi = \zeta \oplus \epsilon \oplus \lambda \oplus \lambda \oplus \lambda$ is also algebraic. Since $w_1(\lambda) = w_1(\zeta \oplus \epsilon)$ [21, p. 246], we have $w_1(\xi) = 0$ and, in view of (*j*), $w_2(\xi) = v$. Thus the proof is complete. □

REFERENCES

[1] AKBULUT, S. and H. KING. A relative Nash theorem. *Trans. Amer. Math. Soc. 267* (1981), 465–481.

[2] AKBULUT, S. and H. KING. The topology of real algebraic sets with isolated singularities. *Ann. of Math. (2) 113* (1981), 425–446.

[3] AKBULUT, S. and H. KING. The topology of real algebraic sets. *L'Enseignement Math. (2) 29* (1983), 221–261.

[4] AKBULUT, S. and H. KING. Polynomial equations of immersed surfaces. *Pacific J. Math. 131* (1988), 209–217.

[5] AKBULUT, S. and H. KING. *Topology of Real Algebraic Sets. Mathematical Sciences Research Institute Publications, 25.* Springer-Verlag, New York, 1992.

[6] BASS, H. *Algebraic K-Theory.* Benjamin, New York, 1968.

[7] BENEDETTI, R. and M. DEDÒ. Counterexamples to representing homology classes by real algebraic subvarities up to homeomorphism. *Compositio Math. 53* (1984), 143–151.

[8] BENEDETTI, R. and J.-J. RISLER. *Real Algebraic and Semialgebraic Sets.* Hermann, Paris, 1990.

[9] BENEDETTI, R. and A. TOGNOLI. On real algebraic vector bundles. *Bull. Sci. Math. (2) 104* (1980), 89–112.

[10] BENEDETTI, R. and A. TOGNOLI. Remarks and counterexamples in the theory of real algebraic vector bundles and cycles. In: Lecture Notes in Math. *959*, 198–211. Springer, Berlin, 1982.

[11] BOCHNAK, J., M. COSTE and M.-F. ROY. *Real Algebraic Geometry.* Ergeb. Math. Grenzgeb. (3), Vol. 36. Springer, Berlin – Heidelberg – New York, 1998.

[12] BOCHNAK, J. and W. KUCHARZ. K-theory of real algebraic surfaces and threefolds. *Math. Proc. Cambridge Philos. Soc. 106* (1989), 471–480.

[13] BOCHNAK, J. and W. KUCHARZ. Algebraic cycles and approximation theorems in real algebraic geometry. *Trans. Amer. Math. Soc. 337* (1993), 463–472.

[14] BOCHNAK, J. and W. KUCHARZ. *On Homology Classes Represented by Real Algebraic Varieties.* Banach Center Publications, Vol. 44, 21–35. Warsaw, 1998.

[15] BOREL, A. and A. HAEFLIGER. La classe d'homologie fondamentale d'un espace analytique. *Bull. Soc. Math. France 89* (1961), 461–513.

[16] DOLD, A. *Lectures on Algebraic Topology.* Springer, Berlin – Heidelberg, 1995.

[17] FULTON, W. *Intersection Theory.* Ergeb. Math. Grenzgeb. (3), Vol. 2. Springer, Berlin – Heidelberg – New York, 1984.

[18] GROTHENDIECK, A. La théorie des classes de Chern. *Bull. Soc. Math. France 86* (1958), 137–154.

[19] HIRONAKA, H. Smoothing of algebraic cycles in low dimensions. *Amer. J. Math. 90* (1968), 1–54.

[20] HIRSCH, M. *Differential Topology.* Graduate Texts in Math., Vol. 33. Springer, New York – Berlin – Heidelberg, 1997.

[21] HUSEMOLLER, D. *Fibre Bundles*, 2nd edition. Springer, New York – Berlin – Heidelberg, 1975.

[22] KUCHARZ, W. Algebraic cycles and algebraic models of smooth manifolds. *J. Algebraic Geom. 11* (2002), 101–127.

[23] KUNZ, E. *Introduction to Commutative Algebra and Algebraic Geometry.* Birkhäuser, Boston – Basel – Stuttgart, 1985.

[24] MILNOR, J. and J. STASHEFF. *Characteristic Classes.* Princeton Univ. Press, Princeton, 1974.

[25] NASH, J. Real algebraic manifolds. *Ann. of Math. (2) 56* (1952), 405–421.

[26] SPANIER, E. *Algebraic Topology.* McGraw-Hill, New York, 1966.

[27] TEICHNER, P. 6-dimensional manifolds without totally algebraic homology. *Proc. Amer. Math. Soc. 123* (1995), 2909–2914.

[28] TOGNOLI, A. Su una congettura di Nash. *Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat. (3) 27* (1973), 167–185.

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