L'Enseignement Mathématique, t. 48 (2002), p. 259–262

AN HOMOLOGY 4-SPHERE GROUP WITH NEGATIVE DEFICIENCY

by Jonathan A. HILLMAN

ABSTRACT. We give an example to show that homology 4-sphere groups need not have deficiency 0.

The *deficiency* def(G) of a finitely presentable group G is the maximum over all finite presentations $\mathcal{P}$ for G of the differences $g-r$, where g is the number of generators and r is the number of relations in the presentation. It is well-known that def(G) may be bounded above by homological invariants [Ep61]. In high dimensions, whether a finitely presentable group can be realized as the fundamental group of an n-manifold with prescribed homology depends only on the homology of the group; in low dimensions ($n \leq 4$) such conditions remain necessary, while constraints on the deficiency often suffice. However bridging the gap between homologically necessary conditions and combinatorially sufficient conditions is usually a delicate matter. This note considers one such situation.

A group G is *perfect* if it is equal to its commutator subgroup G', i.e., if the abelianization $G/G' \cong H_1(G;\mathbf{Z})$ is trivial. If G is the fundamental group of an homology n-sphere then it is finitely presentable and *superperfect*, i.e., $H_1(G;\mathbf{Z}) = H_2(G;\mathbf{Z}) = 0$. These conditions characterize homology n-sphere groups for $n \geq 5$ [Ke69], but in low dimensions more stringent conditions hold. Every perfect group with a presentation of deficiency 0 is an homology 4-sphere group (and therefore is superperfect) [Ke69], but there are finite superperfect groups which are not homology 4-sphere groups [HW85]. As any closed 3-manifold has a handlebody structure with one 0-handle and equal numbers of 1- and 2-handles, homology 3-sphere groups have deficiency 0. However although the finite groups $\mathrm{SL}(2,\mathbf{F}_p)$ are perfect and have deficiency 0 for each prime $p \geq 5$ [CR80] the binary icosahedral group $I^* = \mathrm{SL}(2,\mathbf{F}_5)$ is the only finite homology 3-sphere group.